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# Theory and Algorithms in Graph Rigidity and Algebraic Statistics

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# Theory and Algorithms in Graph Rigidity and Algebraic Statistics

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**Abstract**

The main objective of the proposed meeting is to identify and explore key emerging connections between two active algorithmic research areas: graph rigidity theory, with applications in diverse geometric modeling scenarios and algebraic statistics, with applications in data-driven inference and machine learning. To this end, the invited experts in each field will outline recent fruitful interactions as well as mathematical and computational tools for the fundamental problems in these two areas, to establish a common language for participants with different expertise.

## Background and introduction

### Graph rigidity theory and algorithmic characterizations

A fundamental topic in computational geometry is the efficient algorithmic characterization of basic properties of point configurations in Euclidean space that satisfy various algebraic geometric constraints. Example properties of such feasible sets of point configurations are emptiness, finiteness, singularity etc. Rigidity theory establishes an underlying mathematical foundation by revealing and studying discrete structures, both geometric and purely combinatorial, inherent in geometric constraint systems. This is a classical topic in discrete geometry, whose history can be traced back to the work of Euler, Cauchy and Maxwell on the rigidity of polyhedra and skeletal frames. In the last two decades, both geometric and combinatorial rigidity theory have become particularly active, drawing on diverse areas of mathematics and computer science and engaging with a growing range of modern applications, such as Computer-Aided-Design, molecular modeling, localisation in sensor networks, and the distributed control of formations of autonomous agents. As a result, the area has gained significant international visibility as an active and interdisciplinary research area. In Spring 2021 the Fields Institute in Toronto, Canada, ran a 6-month thematic program on this topic (Geometric Constraint Systems, Framework Rigidity, and Distance Geometry) and an additional 2-month program will take place in Summer 2023. Furthermore, a 6-month program is planned at ICERM, Brown, USA, in Spring 2025 (Geometry of Materials, Packings and Rigid Frameworks).

## Algebraic statistics, emerging connections, timeliness

Recently it has become increasingly clear that rigidity theory and the similarly flourishing field of algebraic statistics – which employs algebraic approaches to data-driven, statistical and machine learning questions – share common goals in capturing hidden discrete structures inherent in algebraic relations. Techniques in one field have been shown to shed a new light on a challenging problem in the other field. One representative example is the estimation of the probability distribution for a sample of observations from a problem domain. Maximum likelihood estimation (MLE) is a common framework used throughout the field of machine learning. MLE involves maximizing a likelihood function in order to find the probability distribution and parameters that best explain the observed data, and hence, provides a framework for predictive modeling in machine learning where finding model parameters can be formulated as an optimization problem. Recently, rigidity theory has been shown to provide useful tools in the realm of Bayesian statistics and Gaussian graphical models. For example, it is shown that the maximum likelihood threshold of a graph, which is the smallest number of data points that guarantees that MLE exist almost surely in the Gaussian graphical model, is closely connected to rigidity theory.

In view of this emerging interaction, it is timely to hold an international workshop in 2024 whose focus and objective is to identify and explore key connections between rigidity theory and algebraic statistics. To this end, the invited experts in each field will outline recent mathematical and computational tools for the fundamental problems in graph rigidity or algebraic statistical inference and modeling, to establish a common language for participants with different expertise.

## Problems in the overlap area of algebraic matroids

A key mathematical concept for facilitating such an interaction would be the theory of algebraic matroids. Matroid theory offers a general framework for analyzing discrete structures and developing efficient algorithms in various fields such as combinatorial optimization and machine learning. Several key questions in rigidity theory and algebraic statistics are inter-related through formulations in terms of matroids associated with algebraic varieties. Thus the workshop aims to provide an occasion for researchers to share the state-of-art techniques for analyzing algebraic matroids and discussing future research directions. Specific problems we would like to address during the workshop are the following.

- Combinatorial and efficient algorithmic characterization of independence/dependence in a variety of algebraic matroids: efficient algorithms will be sought for e.g. checking consistency or redundancy of geometric constraint systems or algebraic relations that capture specific data properties.
- Geometric and efficient algorithmic characterization of the identifiability of algebraic systems in terms of independence/dependence in

algebraic matroids: efficient algorithms will be sought for e.g. checking completeness of algebraic constraint systems and the identifiability of data properties from partial measurements from experiments.

- Development of a common framework for identifying underlying algebraic matroids in applications: this contributes to an acceleration of applications and a systematization of the theory of algebraic matroids.

# Overview of the Meeting

We hosted 25 participants with diverse expertise who engaged in discussions on various aspects of rigidity theory, encompassing both theoretical and algorithmic perspectives, as well as its connections to algebraic statistics, persistent homology, and tropical geometry. Each participant delivered a lightning talk, while five domain experts—Louis Theran, Atsuhiro Nakamoto, Eran Nevo, Tibor Jordan, and Satoshi Murai—gave in-depth one-hour overview lectures. Additionally, Eleftherios Kastis, Jan Legerský, Alison La Porta, Bill Jackson, Dániel Garamvölgyi, Oliver Clarke, and Bernd Schulze presented 30-minute talks.

Jessica Sidman offered insights into her forthcoming book on rigidity theory, tailored for undergraduate and project students, while Jan Legerský demonstrated a new computational platform for solving rigidity-related problems, highlighting examples and novel features. Each talk was followed by an interactive Q&A session and further discussions, with ample time allocated for collaborative working groups over the five days of the seminar.

The social event featured visits to Jomyoji and Hokokuji Temples, accompanied by a traditional Japanese tea ceremony.

# Overview of Talks

## Quad-dominated maps on the sphere (and other surfaces) and their rigidity properties

Brigitte Servatius, WPI

Abstract. Maxwell/Cremona showed that a planar framework can be lifted to a polyhedron. Planar rigidity cycles have a stress that is nonzero on all edges. Their geometric dual is a rigidity cycle as well and therefore the average valence as well as the average face size of such a graph is just a bit less than four and in the absence of very large faces the resulting polyhedron will be quad-dominated. We suggest a simple procedure to easily construct such graphs answering a question of structural engineer Bill Baker.

## NAC-colourings

Anthony Nixon, Lancaster

Abstract. A NAC-colouring is a type of edge-colouring whose existence characterises those graphs that admit a flexible realisation in the plane.

In this talk a positive solution was presented to a conjecture on precisely when a minimally rigid graph in the plane has a NAC-colouring. This represented joint work with Katie Clinch, Daniel Garamvolgyi, John Haslegrave, Tony Huyn and Jan Legerský.

## Generation of quadrangulations on surfaces and related topics

Atsuhiro Nakamoto, Yokohama National University

Abstract. A *quadrangulation* on a surface is a simple graph on the surface such that each face is bounded by a cycle of length 4. A *face contraction* of a face $f$ of $G$ is to identify two opposite vertices of the boundary 4-cycle of $f$ and replace two pairs of multiple edges with two single edges respectively. A *generating theorem* of quadrangulations on the sphere states that every quadrangulation on the sphere can be reduced to a cycle of length 4 by a repeated application of face contractions, through quadrangulations. In my talk, we introduce various generating theorems of quadrangulations on surfaces, and ones with some combinatorial conditions, together with generating theorems of triangulations and Eulerian triangulations.

Finally, focusing on the relation that every quadrangulation on the sphere is a bipartite $(2, 4)$-tight graph, we introduce a generating theorem of bipartite $(2, 2)$-tight graphs, which is related to a bipartite quadrangulation on the projective plane.

## Rigidity over the rational numbers

Sean Dewar, University of Bristol

Abstract. Let us say that a rational framework $(G, p)$ – a pair made up of a simple graph $G = (V, E)$ and a realisation $p : V \to \mathbb{Q}^d$ – is rationally rigid if there are finitely many frameworks $(G, q)$ (modulo isometries) whereby $\|p_v - p_w\| = \|q_v - q_w\|$. With this, we say that a graph $G$ is rigid (resp., flexible) in $\mathbb{Q}^d$ if there exists an open dense subset of rationally rigid (resp., not rationally rigid) realisations of $G$ in $(\mathbb{Q}^d)^V$. If a graph is rigid in $\mathbb{R}^d$, it is easy to show that it is also rigid in $\mathbb{Q}^d$. The reverse, however, gets a little weird. If we take a graph $G$ with 1 degree of freedom in $\mathbb{R}^d$ (i.e., $G$ is not rigid in $\mathbb{R}^d$ but $G + e$ is for some non-edge $e$), then its rigidity in $\mathbb{Q}^d$ is dependent on the genus of a generic realisation's configuration space (the algebraic curve of edge-length equivalent realisations in $(\mathbb{C}^d)^V$ modulo isometries), which now say is the $d$-genus of the graph. Specifically, if $G$ has 1 d.o.f. in $\mathbb{R}^d$, then $G$ is rigid in $\mathbb{Q}^d$ if its $d$-genus is 2 or more as a consequence of Falting's theorem [10, 11], a deep result that is central in Algebraic Number Theory. Furthermore, $G$ is flexible in $\mathbb{Q}^d$ if its $d$-genus is 0. If the $d$-genus is 1, it is entirely unclear what happens.

## The first-order flexibility of a periodic framework

Eleftherios Kastis, Lancaster University

Abstract. Given a periodic placement $p$ of a $\mathbb{Z}^d$-symmetric graph $G$, the rigidity matrix of the resulting framework $\mathcal{G}$ is unitarily equivalent to a multiplication operator $\psi_{\mathcal{G}}$. This operator generalizes the notion of the orbit matrix, providing a route to factor periodic flexes. Moreover, it gives

rise to the $\mathbb{C}(z)$-module $M(\mathcal{G})$ generated by its rows, so the infinitesimal flex space $F(\mathcal{G}, \mathbb{C})$ of $\mathcal{G}$ can be identified with the dual annihilator of $M(\mathcal{G})$. Using spectral synthesis tools (Lefrank, 1958), it was shown that $F(\mathcal{G}, \mathbb{C})$ contains a dense subset of pg-sequences $u_{\omega,h} : \mathbb{Z}^d \to \mathbb{C}^{d\,V_0}$ of the form

$$u_{\omega,h} : k \to \omega^k h(k)$$

where $\omega \in \mathbb{C}_*^d$ lies in the so-called geometric flex spectrum of $\mathcal{G}$ and $h(z)$ is a vector-valued polynomial. A natural question is to extend the above results to $\Gamma$-symmetric frameworks, where $\Gamma$ is a finitely generated abelian group. This is joint work with S. C. Power.

## Almost periodic rigidity for symmetric frameworks

Eleftherios Kastis, Lancaster University

Abstract. Let $\Gamma$ be an abelian group. Given a $\Gamma$-symmetric graph $G$, we can define a pair $(G_0, m)$, where $G_0 = (V_0, E_0)$ be a finite directed multigraph and $m : E_0 \to \Gamma$ is the so-called gain map that encodes the properties of $G$. A $\Gamma$-gain framework is a tuple $\mathcal{G} = (G_0, m, \varphi, \tau)$, where $(G_0, m)$ is a $\Gamma$-gain graph and $\varphi = (\varphi_e)_{e\in E_0}$ is a collection of linear maps from $X$ to $Y$ and $\tau : \Gamma \to \mathrm{Isom}(X)$ is a group homomorphism.

In the last 15 years, there have been studies about factor periodic rigidity (see Power - Owen, 2011). One complication for the study of these frameworks is that the set of factor periodic flexes does not form a linear space, in fact addition of two factor periodic flexes may fail to be factor periodic. Since this set of flexes lives in the kernel of the matrix valued function, it is natural to consider a closed vector space generated by this set. working with the supremum norm, the theory of almost periodic rigidity becomes relative. Almost periodic flexes have been discussed for periodic frameworks (Badri, Kitson, Power, 2014), where it was shown that almost periodic rigidity is equivalent to factor periodic rigidity. Using techniques from abstract analysis we can show that the same statement is true for abstract abelian groups. To prove the result, we use the factorization of the coboundary matrix (see Kastis, Kitson, McCarthy, 2021) and apply theory of joint eigenvalues to give a characterization of the so-called trivial RUM spectrum.

A typical question on the above results is to consider the flex spaces generated under weaker topologies. This is ongoing joint work with Derek Kitson.

## Self-stress, Je T'aime

Oleg Karpenkov, University of Liverpool

Abstract. The title of the talk is inspired by the movie "Paris, Je T'aime". Besides a personal author's attitude to the research subject, the talk follows the structure of the movie: it contains several subject that seems not to be connected in one story but provides a collection of complementary images]. Let us start with the first subject.

**Liftings in $\mathbb{R}^3$.** The notion of Maxwell-Cremona liftings is a classical notion in rigidity theory for frameworks in the plane. The lifting of a self-stress is a certain piecewise linear function whose singular set coincides with the framework. It is natural to ask what happens with the liftings if the framework is not planar. However the answer for planar graphs has been known for more 100 years, a satisfactory definition in the non-planar case was missing.

The direct generalisation here are piece-wise linear functions in $\mathbb{R}^4$ whose singular set is the framework. Unfortunately this approach does not work by the following reason: the singular set of piecewise-linear framework is a two-dimensional polyhedron, while the frameworks are one-dimensional. (Such extension would work perfectly for the case of polyhedral two-dimensional surfaces, introduced by Rybnikov but not for graph frameworks.)

In our recent preprint we proposed to consider liftings to be certain function on the classes of the fundamental group of the complement to the framework. Such functions naturally enumerate the self-stresses (that are interpreted as certain linking numbers). This is a joint research with Fatemeh Mohammadi, Christian Müller, and Bernd Schulze [18, 17].

**Pseudo-periodic tensegrities.** The second subject is on new class of self-stresses in $\mathbb{R}^2$. Such self-stresses admit the following properties:

- They are combinatorially equivalent to a periodic graph on a torus $T^2$.
- Their stresses are all rational. Moreover they are defined by a certain rather simple recurrent formula.
- Their limiting set is a triangle.

Such self-stressed frameworks arise from the study of multidimensional continued fractions of cubic irrationalities. It is interesting to admit, that the set of edges of such frameworks converges to the edges of the triangle. We do not know what is the limiting sets of the vertices of such framework. The vertices of the triangle are the limiting points, however it is not known to the authors if the points of the edges of the triangle are limiting points of vertices.

It is also not known if it is possible to construct similar tensegrities whose limit sets are arbitrary graphs. The question is open even for the case of $K_{3,3}$.

We show a few examples of such self-stresses in the presentation. This is a joint research in progress with Fatemeh Mohammadi, Christian Müller, and Bernd Schulze.

**Rigidity in a curved space.** As a rule self-stresses are considered as a force-loads for a certain finite system of points in the equilibrium. Here it is assumed that the force-loads are geodesics. We propose an alternative approach to self-stresses that covers the situation of curved spaces where the geodesics are not necessarily straight lines in the space. The main idea is as follows. Consider a single point, and let us assume that force levels are certain curves in the plane (or surfaces in the space). In case of circles we have classical tensegrities, however one can consider employ ellipses or more complicated curves. (For instance, one can consider the level sets of the magnetic field that are not centrally symmetric).

So for every point $p$ we have its own family of force levels $f_p(\lambda)$. Now the lines of forces between points $p$ and $q$ will be the curves connecting $p$ and $q$ that satisfy the equation

$$\mathrm{grad}\,(\mathrm{f_p}) = \mathrm{grad}\,(\mathrm{f_q})$$

at all the points of such line of force. Note that such curves are not necessarily parallel to gradients. In addition such curves can have branching points. Once the force lines at a point $p$ are constructed, one can write down the equilibrium conditions in the tangent space at $p$. The next step of this study is to develop a systematic study of such self stresses and the corresponding aspects of the rigidity theory.

## Rigidity and reconstruction in matroids of highly connected graphs

Dániel Garamvölgyi, Alfréd Rényi Institute, Budapest

Abstract. A *graph matroid family* $\mathcal{M}$ is a family of matroids $\mathcal{M}(G)$ defined on the edge set of each finite graph $G$ in a compatible and isomorphism-invariant way. We say that $\mathcal{M}$ has the *Whitney property* if there is a constant $c$ such that every $c$-connected graph $G$ is uniquely determined by $\mathcal{M}(G)$. Similarly, $\mathcal{M}$ has the *Lovász-Yemini property* if there is a constant $c$ such that for every $c$-connected graph $G$, $\mathcal{M}(G)$ has maximal rank among graphs on the same number of vertices.

In the talk, I describe some new results related to these notions. It turns out that if $\mathcal{M}$ is unbounded (that is, there is no absolute constant that bounds the rank of $\mathcal{M}(G)$ for every $G$), then $\mathcal{M}$ has the Whitney property if and only if it has the Lovász-Yemini property, and that every 1-extendable graph matroid family has the Lovász-Yemini (and thus the Whitney) property. These results unify and extend earlier results about graph reconstruction from an underlying matroid.

## Stress-linked pairs of vertices

Dániel Garamvölgyi, Alfréd Rényi Institute, Budapest

Abstract. Let $G$ be a graph. A pair of vertices $\{u, v\}$ is *globally $d$-linked in $G$* if for every generic $d$-dimensional framework $(G, p)$, the edge lengths of $(G, p)$ uniquely determine the distance of $p(u)$ and $p(v)$. In this talk, I introduce a new equilibrium stress-based sufficient condition for being globally $d$-linked and describe how this new notion led to a resolution of some conjectures in combinatorial rigidity theory.

## Computing Tropical Varieties

Fatemeh Mohammadi, KU Leuven and University of Tromsø

Abstract. Tropical geometry has recently found new applications in rigidity theory, particularly in the study of Cayley-Menger varieties, determining bounds on the number of realizations of a graph, and in NAC-coloring. However, computing tropical varieties remains a challenging task, with most progress limited to low dimensions (e.g., $d = 2$) or small-scale examples. Despite the potential of these connections, the computational complexity associated with tropical varieties restricts practical calculations to small instances.

The goal of this presentation is to demonstrate that while computing the entire tropical variety may be infeasible, there are several algebraic and geometric techniques that allow us to sample points from a tropical variety and gain partial information. These techniques rely on the notion

of Gröbner degeneration. More precisely, the concept of the Gröbner fan for a polynomial ideal, pioneered by Mora and Robbiano in 1988, provides a powerful polyhedral framework where the maximal cones correspond to the reduced Gröbner bases of the ideal. The tropical variety emerges as a subcomplex of this Gröbner fan, offering a rich geometric structure used in diverse applications within mathematics and beyond.

In this presentation, we will explore prototypical examples, with a particular focus on tropical Grassmannians and flag varieties. These examples, along with their combinatorial counterparts such as Young tableaux and Gelfand-Cetlin polytopes, provide valuable insights into the broader landscape of tropical geometry. Additionally, we will discuss a geometric approach using polytope mutations, which facilitates Gröbner walks within the tropical fan. This method effectively computes various cones within the tropical varieties.

One specific application of this approach is in computing toric degenerations of varieties—objects of central interest in algebraic geometry that can be modeled using polytopes and polyhedral fans. This approach leverages the well-known correspondence between geometric properties of varieties and combinatorial invariants of their associated polytopes. Through Gröbner fans and tropical geometry, we extend this "dictionary" from toric varieties to more general varieties. I will discuss how to achieve these degenerations using joint work with Oliver Clarke [3].

## On the existence of reflection symmetric flexible realizations

Jan Legerský, Czech Technical University in Prague, Faculty of Information Technology, Czech Republic

Abstract. The existence of a flexible quasi-injective realization in the plane is characterized by the existence of a NAC-coloring [13], which is a surjective coloring of edges by red and blue such that every cycle is either monochromatic, or there are at least two red and at least two blue edges. The idea of NAC-colorings was adjusted to the rotation symmetric setting: there is a rotation symmetric flexible realization if and only if there is a NAC-coloring invariant under the rotation with a certain property [9].

The existence of a reflection symmetric quasi-injective realization with a flex preserving the symmetry, which is the topic of this talk, is surprisingly more difficult. We introduce the concept of pseudo-RS-colorings: an edge coloring by red, blue and gold such that there is at least one blue and one red edge, changing all gold edges to red, resp. all to blue, yields NAC-colorings and blue and red interchange under the reflection. An almost red-blue cycle is a cycle that has exactly one gold edge. A pseudo-RS-coloring is an RS-coloring either if there is no almost red-blue cycle, or for every red-blue cycle, there is another pseudo-RS-coloring differing in a specific way on the cycle.

Our main results are the following: if a graph admits a reflection symmetric flexible quasi-injective realization, then the graph has an RS-coloring. This necessary condition can be strengthened to exclude some RS-colorings that cannot come from a flex. On the other hand, we show that if a graph has an RS-coloring with no almost red-blue cycle, then it has a reflection symmetric flexible quasi-injective realization. There is also a construction of a reflection symmetric flex in a very special case of RS-

colorings with an almost red-blue cycle, but the complete characterization is still open.

This is joint work with Sean Dewar and Georg Grasegger.

## PyRigi - software package for rigidity theory

Jan Legerský, Czech Technical University in Prague, Faculty of Information Technology, Czech Republic

Abstract. PyRigi is a Python package for research in rigidity and flexibility of bar-and-joint frameworks that was initiated at the workshop *Code of Rigidity* held during the Special Semester on Rigidity and Flexibility at RICAM in Linz, Austria in March 2024. In this talk, we discuss the current status of the package including the documentation, communication tools and possible ways of contributing. As an example of using the package we present a generically rigid graph with two different penny realizations which is a solution to an open problem in [7].

## A smallest flexible polyhedron without self-intersections

Georg Grasegger, Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences

Abstract. For some time it was believed that Steffen's polyhedron is the smallest flexible polyhedron without self-intersection. In this talk we show a smaller example with 8 vertices which is now known to be the least possible.
Joint work with Matteo Gallet, Jan Legerský and Josef Schicho

## $k$-fold circuits in rigidity matroids

Anthony Nixon, Lancaster University

Abstract. The concept of a $k$-fold circuit generalises double circuits to arbitrary cyclic sets in a matroid with the dependencies parametrised by $k$. This talk analysed some basic theory of $k$-fold circuits and used it to analyse extensions of the well known coning lemmas in rigidity theory. Then a further application resolved a conjecture of Alan Lew by proving the maximal dimension in which the graph obtained from $K_{2d}$ by deleting a perfect matching is rigid.
This talk concerned joint work with John Hewetson, Bill Jackson and Ben Smith.

## Newton-Okounkov Bodies and Algebraic Matroids

Oliver Clarke, Edinburgh

Abstract. A *Newton-Okounkov body* $\Delta(A, \nu)$ is a closed covex body associated to a finitely generated algebra $A$ and valuation $\nu : A \setminus \{0\} \to \mathbb{Q}^n$. A finite *Khovanskii basis* is a set of generators $f_1, \ldots, f_k$ for $A$ such that their images under $\nu$ generate the semigroup $\nu(A)$. These generating sets are analogous to Gröbner bases and provide straightforward algorithmic methods to solve questions such as the membership problem. Unlike Gröbner bases, finite Khovanskii bases need not exist since the semigroup

$\nu(A)$ may not be finitely generated. However, for special families, such as Grassmannians, it is possible to construct collections of well-behaved weight-valuations.

Given an algebra $A$ with distinguished generating set $F = \{f_1, \dots, f_k\} \subseteq A$, its *algebraic matroid* $M$ is the matroid on ground set $F$ whose dependent sets are the subsets $\{d_1, \dots, d_\ell\} \subseteq F$ such that there exists a non-zero polynomial $p \in \mathbb{C}[y_1, \dots, y_\ell]$ with $p(d_1, \dots, d_\ell) = 0$. If $f_1, \dots, f_k$ is a Khovanskii basis for some valuation $\nu$, then the matroid $M_\nu$ realised by vectors $\nu(f_1), \dots, \nu(f_k)$ is closely related to $M$. Moreover, for special families, it is possible to recover fully the data of $M$ by varying $\nu$. For example, for the Grassmannian of 2-planes, the Plücker coordinates are a Khovanskii basis with respect to any weight valuation, and each base of the algebraic matroid is a base of some matroid $M_\nu$.

## On generic universal rigidity on the line

Tibor Jordán, ELTE Eötvös Loránd University, Budapest

Abstract. A $d$-dimensional bar-and-joint framework $(G, p)$ with underlying graph $G$ is called universally rigid if all realizations of $G$ with the same edge lengths, in all dimensions, are congruent to $(G, p)$. A graph $G$ is said to be generically universally rigid in $\mathbb{R}^d$ if every $d$-dimensional generic framework $(G, p)$ is universally rigid. In this short talk we summarize the main results of a recent paper [6] concerning the case $d = 1$. We gave counterexamples to a conjectured characterization of generically universally rigid graphs in $\mathbb{R}^1$ from (R. Connelly, 2011). We also introduced two new operations that preserve the universal rigidity of generic frameworks, and the property of being not universally rigid, respectively. One of these operations is used in the analysis of one of our examples, while the other operation is applied to obtain a lower bound on the size of generically universally rigid graphs. This bound gives a partial answer to a question from [15].

## Minimally rigid tensegrity frameworks

Tibor Jordán, ELTE Eötvös Loránd University, Budapest

Abstract. A $d$-dimensional tensegrity framework $(T, p)$ is an edge-labeled geometric graph in $\mathbb{R}^d$, which consists of a graph $T = (V, B \cup C \cup S)$ and a map $p : V \to \mathbb{R}^d$. The labels determine whether an edge $uv$ of $T$ corresponds to a fixed length bar in $(T, p)$, or a cable which cannot increase in length, or a strut which cannot decrease in length.

We consider minimally infinitesimally rigid $d$-dimensional tensegrity frameworks and provide tight upper bounds on the number of its edges, in terms of the number of vertices and the dimension $d$. We obtain stronger upper bounds in the case when there are no bars and the framework is in generic position. The proofs use methods from convex geometry and matroid theory. A special case of our results confirms a conjecture of W. Whiteley from 1987. We also give an affirmative answer to a conjecture concerning the number of edges of a graph whose three-dimensional rigidity matroid is minimally connected (Joint work with Adam Clay and Sára Tóth.)

## The infinitesimal rigidity of Dihedral-symmetric bar-joint frameworks

Alison La Porta, Lancaster University

Abstract. The infinitesimal rigidity of symmetric finite plane bar-joint frameworks has been researched extensively for over ten years. A number of combinatorial characterisations of such frameworks were established, provided the frameworks are 'symmetry generic'. If the symmetry group which acts on a given framework is dihedral, the problem becomes more difficult, especially if the symmetry group does not act freely on the joints. One of the difficulties arises from the fact that the *gain graph*, a combinatorial tool often used in the study of infinitesimal rigidity of symmetric frameworks, loses some useful properties in this setting.

## Forced symmetric rigidity on the plane

Alison La Porta, Lancaster University

Abstract. Take a finite plane bar-joint framework $(\tilde{G}, \tilde{p})$ with a non-trivial symmetry, and consider the infinitesimal motions which do not break its symmetry. If all such motions are trivial, we say $(\tilde{G}, \tilde{p})$ is fully-symmetrically (infinitesimally) rigid. In 2012, Jordán, Kaszanitzky and Tanigawa combinatorially characterised fully-symmetrically infinitesimally rigid frameworks, where the symmetry group is either a finite cyclic group or a Dihedral group $\mathcal{C}_{kv}$ for some odd $k \geq 3$, provided the symmetry group acts freely on the vertex set and the framework is 'symmetry-generic'. I present similar results, for which I drop the requirement that the group action is free on the joints.

Consider the quotient $G$ of the underlying graph $\tilde{G}$, and suppose that the symmetry group acts freely on $V(\tilde{G})$. The edges of $G$ may be directed and labelled with group elements of the symmetry group, in a specific way which allows to maintain all of the information of $\tilde{G}$, while disregarding any redundancy. The combinatorial tool obtained, typically referred to as a *gain graph*, is often denoted $(G, \psi)$, where $\psi : E(G) \to \Gamma$ denotes the labelling of the edges. If the symmetry group does not act freely on $V(\tilde{G})$, then some of the information of $\tilde{G}$ is lost through the process of obtaining $(G, \psi)$. Namely, $(G, \psi)$ cannot store any information on the stabilisers of the vertices. Hence, I introduce a generalisation of gain graph which also labels vertices with stabiliser groups. Some examples show that this new gain graph does not carry all properties of the usual gain graph.

## Maxwell-Cremona liftings of self-stressed frameworks in arbitrary dimension

Bernd Schulze, Lancaster University

Abstract. This is an extended talk on the first topic mentioned in Oleg Karpenkov's lightning talk.

In 1864, James Clerk Maxwell introduced a link between self-stressed frameworks in the plane and piecewise linear liftings to 3-space. This connection has found numerous applications in areas such as rigidity theory, discrete and computational geometry, control theory and structural engineering. While there are some generalisations of this theory to liftings of

d-complexes in d-space, extensions for liftings of frameworks in d-space for d at least 3 have been missing. In this talk we introduce differential liftings on general graphs using differential forms associated with the elements of the homotopy groups of the complements to the frameworks. Such liftings play the role of integrands for the classical notion of liftings for planar frameworks. These differential liftings have a natural extension to self-stressed frameworks in higher dimensions. As a result we generalise the notion of classical liftings to both graphs and multidimensional k-complexes in d-space (k=2,. . . ,d).

This is joint work with Oleg Karpenkov, Fatemeh Mohammadi and Christian Mueller.

## Inverse problems in topological data analysis

Louis Theran, University of St Andrews

Abstract. Persistent homology, which captures how the topology of a data set changes across scales, is a basic tool in topological data analysis. The main invariant of persistent homology, called the *barcode*, is a collection of intervals corresponding to lifetimes of homological features in a filtration indexed by a subset of $\mathbb{R}$. Forward problems in persistent homology are well-studied and foundational results imply that the map from a data set to its barcode is, in wide generality, stable: similar data sets have similar barcodes.

Much less is known about inverse problems in persistent homology. In this talk, I will consider point cloud data and the Vietoris–Rips and Čech filtrations, which are the most commonly used the applications. I will describe combinatorial sufficient conditions for a generic point cloud to be locally identifiable, up to isometry, from its barcode in either filtration and identifiable from its barcode in the Vietoris–Rips filtration. The identifiability result makes use of recent advances in generic unlabelled global rigidity.

This is joint work with D. Beers, H. Harrington, J. Leygoine, and U. Lim.

## Volume Constrained Rigidity of Hypergraphs

Bill Jackson, Queen Mary University of London

Abstract. Let $H = (V, E)$ be a hypergraph and $p : V \to \mathbb{R}^d$ be a realisation of $H$ in $\mathbb{R}^d$. We say that $(H, p)$ is *volume rigid* if every continuous motion of the vertices of $(H, p)$ which preserves the volume of the hyperedges in $E$ results in a realisation which is congruent to $p$. The hypergraph $H$ is *volume rigid in* $\mathbb{R}^d$ if some (or equivalently, every) generic realisation of $H$ in $\mathbb{R}^d$ is volume rigid. We prove the following result (which extends previous results of Kalai and Fogelsanger for the special case when $j = 1$).

**Theorem.** Suppose $\Delta$ is a simplicial $d$-manifold and $H$ is the hypergraph defined by the $j$-faces of $H$ for some fixed $1 \leq j \leq d-1$. Then $H$ is volume rigid in $\mathbb{R}^{d+1}$.

This is joint work with James Cruickshank and Shin-ichi Tanigawa.

## List of Participants

- Katie Clinch (University of New South Wales, Australia)
- Oliver Clarke (University of Edinburgh, UK)
- Sean Dewar (University of Bristol, UK)
- Dániel Garamvölgyi (Alfréd Rényi Institute of Mathematics, Hungary)
- Georg Grasegger (Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Austria)
- Yuya Higashikawa (University of Hyogo, Japan)
- Bill Jackson (Queen Mary University of London, UK)
- Tibor Jordán (ELTE Eötvös Loránd University, Hungary)
- Oleg Karpenkov (University of Liverpool, UK)
- Eleftherios Kastis (Lancaster University, UK)
- Viktória Kaszanitzky (Budapest University of Technology and Economics, Hungary)
- Naoki Katoh (University of Hyogo, Japan)
- Csaba Király (ELTE Eötvös Loránd University, Hungary)
- Alison La Porta (Lancaster University, UK)
- Jan Legerský (Czech Technical University, Czech Republic)
- Fatemeh Mohammadi (KU Leuven, Belgium; University of Tromsø, Norway)
- Satoshi Murai (Waseda University, Japan)
- Atsuhiro Nakamoto (Yokohama National University, Japan)
- Eran Nevo (Hebrew University, Israel)
- Anthony Nixon (Lancaster University, UK)
- Brigitte Servatius (Worcester Polytechnic Institute, USA)
- Bernd Schulze (Lancaster University, UK)
- Jessica Sidman (Amherst College, USA)
- Meera Sitharam (University of Florida, USA)
- Shin-ichi Tanigawa (University of Tokyo, Japan)
- Louis Theran (University of St Andrews, UK)

# Meeting Schedule

**Check-in Day: September 1(Sun)**

- Welcome Banquet

**Day1: September 2 (Mon)**

- Short/flash talks
- Talk by Nakamoto
- Short/flash talks
- Discussions

**Day2: September 3 (Tue)**

- Talk by Theran
- Short/flash talks
- Talk by Murai
- Discussions
- Introduction to PyRigi by Legersky

**Day3: September 4 (Wed)**

- Talks by Nixon and Kastis
- Talks by La Porta and Garamvolgyi
- Excursion to Kamakura and Main Banquet

**Day4: September 5 (Thu)**

- Talk by Nevo
- Talks by Clarke and Legersky
- Talk by Jordan
- Discussions

**Day5: September 6 (Fri)**

- Talks by Jackson and Schulze
- Discussions
- Wrap up

# Summary of Discussions

Participants presented their latest research findings and highlighted key challenges, sparking a range of engaging discussions. In particular, several concrete open problems proposed during the meeting prompted participants to share their insights and collaboratively explore potential directions for future research.

## Problems on graph theoretical analysis of non-negative tensor decompositions and completions

(posed by Shin-ichi Tanigawa)

Tensor decompositions and completions are widely used in various context of computer science. For statistical applications, it is required that each entry in the decomposition is non-negative (since probability is non-negative).

Tensor decompositions and completions can be understood in the rigidity frameworks: unlike a conventional rigidity formulation, the underlying graph is a $k$-regular hypergraph $G$ and the measurement map $f_G$ of $G$ is

$$f_G(p) = \left( \sum_{j=1}^{d} \prod_{v \in e} p_{v,i} \right)_{e \in E(G)} \qquad (p \in \mathbb{R}^{dn})$$

where $p_{v,i}$ denotes the $i$-th coordinate of the vertex $v$. The formulation has been discussed in my paper with Cruickshank, Nixon, and Mohammadi, and it may be considered as a hypergraph extension of the matrix completion formulation due to Singer-Cucuringu.

A non-negative matrix decomposition or completion corresponds to the case when each entry of $p$ is non-negative.

Krone and Kubjas analyzed the non-negative matrix completion problem using rigidity. That corresponds to the case when the order of tensors is two and the underlying graphs are complete.

## Problems on tensegrity realizability of Wagner graphs

(posed by Shin-ichi Tanigawa)

The question concerns about the low dimensional realizability of tensegrities.

A *tensegrity* in $\mathbb{R}^d$ is defined as a triple $(G, \sigma, p)$, where $G$ is a multigraph, $\sigma : E(G) \to \{-, +\}$ is a sign function on the edge set of $G$, and $p : V(G) \to \mathbb{R}^d$ is a point configuration. We say that $(G, \sigma, p)$ is *d-dimensional* if the affine span of $p(V(G))$ is $d$-dimensional. We say $(G, \sigma, q)$ is a *deformation* of $(G, \sigma, p)$ if $\sigma(ij)\|q_i - q_j\| \leq \sigma\|p_i - p_j\|$ for every $ij \in E(G)$.

Here is a key definition. A multigraph $G$ is *d-realizable* if for any integer $d' \geq d$ and any $d'$-dimensional tensegrity $(G, \sigma, p)$ there is a $d$-dimensional tensegrity $(G, \sigma, q)$ that is a deformation of $(G, \sigma, p)$. The realizability number of $G$ is defined as the smallest integer $d$ such that $G$ is $d$-realizable.

This is the tensegrity version of the realizability of graphs by Connelly-Belk (DCG2007) and it has been introduced by myself with a joint work with Ryoshun Oba. It is an unsolved open problem to characterize multigraphs with the realizability number at most three. In an on-going work with Tibor and Daniel, we found that the realizability of the Wagner graph is a key. The Wagner graph is a 3-regular connected graph with 8 vertices obtained from the octagon by drawing diagonal edges between opposite vertices.

My open problem is to decide the realizable dimension of the Wagner graph (with possible parallel edges). Since the Wagner graph is just a graph with 8 vertices, it would be possible to check it by a computer. The question is if there is a implementable algorithm for computing the realizability dimension. (I do not expect the algorithm to work in polynomial-time.)

## Problems on absolute 2-rigidity

(recalled by Jan Legerský)

In 1999, Hiroshi Maehara posed the problem to characterize absolutely 2-rigid graphs, where a graph $G$ is called *absolutely $d$-rigid* if every injective realization of $G$ in $\mathbb{R}^d$ is rigid. The question is interesting only for $d = 2$ since a graph is absolutely 1-rigid if and only if it is connected, and absolutely $d$-rigid for $d \geq 3$ if and only if it is complete. The latter statement follows from the fact the a so called butterfly motion come from an injective realization such that two non-adjacent vertices are placed arbitrarily and all the other vertices are placed on a line.

Clearly, a necessary condition on absolute 2-rigidity is 2-rigidity. A sufficient, but not necessary, condition is given by non-existence of NAC-colorings [13]. A stronger sufficient condition can be formulated using the so called *constant distance closure* [14].

Discussions of the problem: Dániel Garamvőlgyi proposed to attempt a proof of NP-hardness. Eran Nevo suggested to look at 2-hyperconnectivity matroid.

## Problems on volume rigidity

(posed by Eran Nevo)

A pure $(d-1)$-dimensional simplicial complex is *volume rigid* if for some (eq. every) generic embedding of its vertices in $\mathbf{R}^{d-1}$, every motion of the vertices that preserves, up to first order, the volumes of its facets, in fact does so for all $d$-subsets of the vertices. In a paper by Bulavka-N.-Peled [2] we conjectured that:

Conj.1: For every compact connected surface without boundary $S$, every triangulation $\Delta$ of $S$ and every facet (triangle) $F \in \Delta$, $\Delta \setminus \{F\}$ is volume rigid (in the plane).

Conj.2: For every $d \geq 3$, every triangulation $\Delta$ of the $(d-1)$-sphere and every facet $F \in \Delta$, $\Delta \setminus \{F\}$ is volume rigid.

Conj.1 is known for the 2-sphere and the torus [2] and open otherwise. (Without removing a triangle, it is known also for the projective plane and the Klein bottle [2].) Conj.2 is open for every $d \geq 4$.

## Problems on circumsphere rigidity

(posed by Louis Theran)

Let $d \geq 1$ and $2 \leq k \leq d+1$. If $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_k)$ is a configuration of affinely independent points in $\mathbb{R}^d$, the minimum radius $r$ of a circumsphere of $\mathbf{p}$ satisfies

$$r^2 = -\frac{\det \Lambda_{\mathbf{p}}}{2 \det \Delta_{\mathbf{p}}}$$

where $\Delta_{\mathbf{p}}$ is the $(k+1) \times (k+1)$ Cayley–Menger matrix of $\mathbf{p}$ and $\Lambda_{\mathbf{p}}$ is obtained from $\Delta_{\mathbf{p}}$ by removing the first row and column.

A circumsphere framework, $(H, \mathbf{p})$ is a pair with $H = ([n], E)$ a hypergraph with $2 \leq |\sigma| \leq d+1$ and $\mathbf{p}$ a configuration of $n$ points in $\mathbb{R}^d$. We define a measurement map

$$\rho_H(\mathbf{p}) = \left( -\frac{\det \Lambda_{\mathbf{p}_\sigma}}{2 \det \Delta_{\mathbf{p}_\sigma}} \right)_{\sigma \in E}$$

As usual, we say that $(H, \mathbf{p})$ is locally rigid if there is a neighborhood $U \ni \mathbf{p}$ so that if $\mathbf{q} \in U$ and $\rho_H(\mathbf{p}) = \rho_H(\mathbf{q})$, then $\mathbf{q}$ is congruent to $\mathbf{p}$.

Circumsphere rigidity arises in some ongoing work with Beers, Harington, Leygoine, and Lim on identifiability problems in persistent homology. We have shown that local circumsphere rigidity is a generic property and that there are locally circumsphere rigid hypergraphs. Beyond this, little is known. Some interesting problems are:

- Let $2 \leq r \leq d+1$. Is the complete $r$-uniform hypergraph $K_n^r$ locally circumsphere rigid when $\binom{n}{r} \geq dn - \binom{d+1}{2}$?
- Let $2 \leq r \leq d+1$. Is the complete $r$-uniform hypergraph $K_n^r$ globally circumsphere rigid when $\binom{n}{r} \geq dn - \binom{d+1}{2}$?
- Sufficient conditions for local circumsphere rigidity, e.g., inductive constructions.

## Problems on uniqueness of degree sequences in base decompositions

(posed by Shin-ichi Tanigawa)

Let $K_n^k$ be the $k$-uniform complete hypergraph on $\{v_1, \dots, v_n\}$, and let $\mathcal{M}$ be a matroid on the edge set of $K_n^k$. For a subgraph $H$ of $K_n^k$, its *(labeled) degree sequence* is defined to be $\deg(H) = (d_H(v_1), \dots, d_H(v_n))$, where $d_H(v_i)$ denotes the degree of $v_i$ in $H$.

Let $G$ be a spanning subgraph of $K_n^k$ whose edge set is the union of edge-disjoint $s$ bases of $\mathcal{M}$. By a *base-decomposition* of $G$, we mean a sequence $(G_1, \dots, G_s)$ of subgraphs of $G$ such that $G_i$'s are mutually edge-disjoint and each $E(G_i)$ is a base of $\mathcal{M}$.

The question we want to understand is for which matroid $\mathcal{M}$ a base-decomposition has the unique degree sequence property. We say that a base-decomposition $(G_1, \dots, G_s)$ has the unique degree sequence property if there is no base-decomposition $(H_1, \dots, H_s)$ satisfying $H_1 \neq G_1$ and $\deg(H_1) = \deg(G_1)$.

We have a proof showing that, if $\mathcal{M}$ is an even cycle matroid of a graph, then there is no base-decomposition having the unique degree sequence property. Our conjecture is that this is also the case for graphic matroids.

## Problems on rigidity matroid for graphs embedded on surfaces via Delta-matroids

(posed by Brigitte Servatius)

Bouchet associated a Delta-matroid for a cellular map on a surface using topology. In [1], we give a combinatorial version. There are two matroids arising from this Delta-matroid, namely the lower matroid whose bases are the feasible sets of minimal size and the upper matroid whose bases are feasible sets of maximal size. The difference in rank of these two matroids is 2 minus the Euler characteristic of the surface. The lower matroid is the cycle matroid of the embedded graph, while the upper matroid is the cocycle matroid of the geometric dual. on the sphere upper and lower matroid are identical and the 2-dimensional rigidity matroid is the Dilworth truncation of two copies of the cycle matroid for a planar graph. For cellular embeddings of a graph on other surfaces we propose to consider the Dilworth truncation of the union of upper and lower matroid as the rigidity matroid of the graph on the surface.

## Problems on infinitesimal rigidity of plane frameworks with dihedral symmetry

(posed by Bernd Schulze)

A well-studied problem in rigidity theory is to combinatorially characterise the graphs that give symmetry-forced rigid frameworks in the plane if realised generically with the given symmetry. (A framework is forced-symmetric rigid if it has no non-trivial symmetry-preserving motions.) A key tool to study forced-symmetric infinitesimal rigidity is the orbit rigidity matrix, which is an analog of the standard rigidity matrix for the forced-symmetric setting and was first introduced by Schulze and Whiteley in 2010. Using these matrices, characterisations of symmetry-generic symmetry-forced rigid frameworks in the plane have been obtained for the reflection group, rotational (cyclic) groups and dihedral groups of order 2k+1, where k is at least 1. See the work of Kaszanitzky, Jordán and Tanigawa (2016) and of Malestein and Theran (2015), for example.

To study whether a symmetry-generic framework is infinitesimally rigid (instead of just symmetry-forced rigid), phase-symmetric orbit rigidity matrices have been established for groups of order 2 and 3 by Schulze and Tanigawa in 2015, and for arbitrary rotational groups by Ikeshita and Tanigawa shortly afterwards. Each of these matrices corresponds to an irreducible representation of the group and allows an analysis of the infinitesimal motions and self-stresses of the symmetry type described by the representation. Using these phase-symmetric orbit rigidity matrices and their underlying combinatorial structure, known as *group-labeled quotient graphs* or *gain graphs*, characterisations for symmetry-generic infinitesimal rigidity have been obtained for a large class of cyclic groups by the above-mentioned authors.

It remains a key challenge to analyse plane framework with dihedral symmetry. In particular, dihedral groups of order larger than 4 are non-abelian and hence have irreducible representations of order larger than 1. It is an open question how to define phase-symmetric orbit rigidity matrices for these representations. A related question is what the underlying combinatorial structures of these matrices look like. An answer to these questions may lead to interesting new types of rigidity matroids.

## Problems on volume constrained rigidity of hypergraphs

(posed by Bill Jackson)

Let $H = (V, E)$ be a hypergraph and $p : V \to \mathbb{R}^d$ be a realisation of $H$ in $\mathbb{R}^d$. We say that $(H, p)$ is *volume rigid* if every continuous motion of the vertices of $(H, p)$ which preserves the volume of the hyperedges in $E$ results in a realisation which is congruent to $p$. The hypergraph $H$ is *volume rigid in* $\mathbb{R}^d$ if some (or equivalently, every) generic realisation of $H$ in $\mathbb{R}^d$ is volume rigid.

**Problem 1.** Suppose $P$ is a convex simplicial $d$-polytope, $H$ is the hypergraph defined by the $j$-faces of $P$ for some fixed $1 \leq j \leq d-2$ and $p$ is the realisation of $H$ in $\mathbb{R}^d$ given by the positions of the vertices of $P$. Is $(H, p)$ volume rigid in $\mathbb{R}^d$? (This is true when $j = 1$ by results of Dehn and Whiteley.)

**Problem 2.** Suppose $H$ is a $j$-uniform hypergraph and $H$ is volume rigid in $\mathbb{R}^{d+1}$ for some $2 \leq j \leq d$. Is $H$ volume rigid in $\mathbb{R}^d$? (We can use Whiteley's Coning Lemma to show this is true when $j = 1$.)

# Challenges and Future Directions

By highlighting recent interactions between rigidity theory and algebraic statistics, the importance of algebraic matroids was reaffirmed. New applications of algebraic matroids, such as volume rigidity and identifiability in persistent homology, were also discussed. However, it was noted that, despite the rapid expansion of application areas for algebraic matroids, there remains a lack of effective tools for analyzing their combinatorial structure. In this context, several specific challenges and future research directions were identified.

- Understanding the relationship between operations on algebraic varieties and operations on matroids: In algebraic geometry, various construction operations on algebraic varieties have been proposed, while in combinatorics, corresponding construction operations on matroids have been developed independently. However, certain operations exhibit striking similarities, and understanding these similarities through the lens of algebraic matroids is expected to shed light on the seemingly complex structure of algebraic matroids. In particular, some of the participants initiated a project exploring the connection between matroid union and secant varieties.
- Analysis of algebraic matroids via tropicalization: Except for the cases of Cayley-Menger varieties and Grassmannians, the analysis of matroid structures through the tropicalization of algebraic varieties has not been carried out, and a more systematic understanding is needed.
- Combinatorics of matroids on hypergraphs: Matroids on graphs are one of the central topics in matroid theory, and various types of graph-based matroids have been proposed, with their relationships and combinatorial properties being well-studied. In contrast, matroids on hypergraphs, which are particularly important in algebraic statistics, seem to lack systematic understanding to date. Some of the participants initiated a project exploring combinatorial characterizations of sparsity matroids of hypergraphs.
- Global rigidity under general measurements and matroid redundancy or connectivity: Global rigidity in Euclidean rigidity theory is characterized by the redundancy or connectivity of the rigidity matroid, and thus, graph conditions ensuring redundancy or connectivity of the rigidity matroid have been extensively studied. However, the relationship between global rigidity under general measurements and the redundancy or connectivity of the corresponding algebraic matroid remains largely unexplored.

**Some of presentations at the workshop are now on arXiv:**

- Eran Nevo published a preprint [19] on his open problem about $k$-volume rigidity of simplicial complexes in $\mathbb{R}^d$ (with Alan Lew, Yuval Peled, Orit Raz).
- Bill Jackson and Shin-ichi Tanigawa (with James Cruickshank) published a preprint [5] on the open problem about volume constrained rigidity of hypergraphs.
- Sean Dewar, Georg Grasegger, and Jan Legerský published a preprint [8] on constructing reflection-symmetric flexible realisations of graphs.
- Oleg Karpenkov, Fatemeh Mohammadi, and Bernd Schulze (with Christian Müller) published a preprint [18] on a generalization of

Maxwell-Cremona lifting to the three-dimensional case and further to multidimensional cases.

- Oleg Karpenkov, Fatemeh Mohammadi, and Bernd Schulze (with Christian Müller) published a preprint [17] on new type of self-stressed tensegrities that admits remarkable Dirichlet periodicity filling the triangle.
- Oleg Karpenkov published a preprint [16] on tensegrities on manifolds.
- Daniel Garamvölgyi published a preprint [12] on the reconstruction of highly connected graphs from matroids.
- Tibor Jordán published a preprint [4] on minimally rigid tensegrity frameworks (with Adam D. W. Clay and Sara H. Toth).
- Fatemeh Mohammadi, Louis Theran, and Jessica Sidman are finalizing a paper titled 'Algebraic Matroids of Secant Varieties,' which originated from the discussions during the meeting.

## References


[1] R. C. Avohou, B. Servatius, and H. Servatius. Maps and $\delta$-matroids revisited. *Art Discrete Appl. Math. 4*, 2021.

[2] D. Bulavka, E. Nevo, and Y. Peled. Volume rigidity and algebraic shifting. arXiv:2211.00574, 2022.

[3] O. Clarke, F. Mohammadi, and F. Zaffalon. Toric degenerations of partial flag varieties and combinatorial mutations of matching field polytopes. *Journal of Algebra*, 638:90–128, 2024.

[4] A. D. W. Clay, T. Jordán, and S. H. Toth. Minimally rigid tensegrity frameworks. arXiv:2410.07452, 2024.

[5] J. Cruickshank, B. Jackson, and S. Tanigawa. Volume rigidity of simplicial manifolds. arXiv:2503.01647, 2025.

[6] G. Z. Dantas e Moura, T. Jordán, and C. Silverman. On generic universal rigidity on the line. arXiv:2305.14027, 2023.

[7] S. Dewar, G. Grasegger, K. Kubjas, F. Mohammadi, and A. Nixon. On the uniqueness of collections of pennies and marbles. arXiv: 2307.03525, 2023.

[8] S. Dewar, G. Grasegger, and J. Legerský. Constructing reflection-symmetric flexible realisations of graphs. arXiv:2408.06928, 2024.

[9] S. Dewar, G. Grasegger, and J. Legerský. Flexible placements of graphs with rotational symmetry. In W. Holderbaum and J. M. Selig, editors, *2nd IMA Conference on Mathematics of Robotics (IMA 2020)*, pages 89–97. Springer International Publishing, 2022.

[10] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Inventiones Mathematicae*, 73(3):349–366, Oct 1983.

[11] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Inventiones Mathematicae*, 75(2):381–381, Jun 1984.

[12] D. Garamvölgyi. Rigidity and reconstruction in matroids of highly connected graphs. arXiv:2410.23431, 2024.

[13] G. Grasegger, J. Legerský, and J. Schicho. Graphs with Flexible Labelings. *Discrete & Computational Geometry*, 62(2):461–480, 2019.

[14] G. Grasegger, J. Legerský, and J. Schicho. Graphs with flexible labelings allowing injective realizations. *Discrete Mathematics*, 343(6):Art. 111713, 2020.

[15] T. Jordán and V. H. Nguyen. On universally rigid frameworks on the line. *Contributions to Discrete Mathematics*, 10(2), 2015.

[16] O. Karpenkov. Tensegrities on the space of generic functions. arXiv:1905.11262, 2019.

[17] O. Karpenkov, F. Mohammadi, and B. S. Christian Müller. Klein-arnold tensegrities. arXiv:2410.12729, 2024.

[18] O. Karpenkov, F. Mohammadi, C. Müller, and B. Schulze. A differential approach to maxwell-cremona liftings. arXiv:2312.09891, 2023.

[19] A. Lew, E. Nevo, Y. Peled, and O. E. Raz. On the $k$-volume rigidity of a simplicial complex in $\mathbb{R}^d. arXiv: 2503.01665, 2025.$