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Markov Chain Monte Carlo 2.0

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Overview of the meeting

Markov chains and the related Monte Carlo method are ubiquitous algorithmic techniques across all scientific areas. Despite significant practical successes, theoretically analyzing the convergence rate (technically known as the mixing rate) remains a challenging mathematical question for many fundamental problems, especially in discrete settings. In theoretical computer science, these problems are studied as computational counting and sampling problems. In recent years, a number of new techniques emerged to tackle long-standing counting and sampling challenges. These techniques include both new ways to analyze Markov chains, and alternative algorithmic approaches to MCMC. In this meeting, we aim to bring together people in these directions to uncover deeper connections of the exciting new advances.

To be more specific, the typical goal is to sample from a complicate distribution, such as the Gibbs distribution, or to estimate the normalizing constant (known as the partition function), marginal probabilities, and other quantities of interest for the said distribution. In most situations, counting and sampling are equivalent for polynomial-time computation. One may approach this goal via probabilistic methods, such as Markov chains and other resampling based techniques, or via algebraic methods, such as Barvinok's interpolation method, especially combined with cluster expansions. One aim of the meeting is to find unifying viewpoints and hidden links between probabilistic approaches and algebraic approaches.

Next we list a few main research topics to be discussed in the workshop.

High dimensional expanders

The notion of High dimensional expanders is a generalisation of expander graphs to simplicial complexes and hypergraphs. One important aspects of this perspective is to make the local-to-global arguments possible, which allows us to analyse random walks over a global structure via analysing the spectral gaps and other properties of local structures. This technique has led to the breakthrough result of the rapid mixing of the bases-exchange walks for matroids.

Resampling methods

Resampling methods provide alternative sampling techniques to Markov chains. In particular, the recent progress of resampling methods has strong connections to the Lovász local lemma, particularly the constructive version of Moser and Tardos. Resampling methods have led to the first polynomial-time approximation algorithm for network reliability, and have other unique features comparing to Markov chain methods. For example, they are usually dynamic and distributed by design.

Zeros of partition functions

Zeros of partition functions are a traditional topic in statistical physics, where the motivation is to study phase transitions via the analyticity of partition functions. More recently, Barvinok has introduced an algorithmic technique which can utilize the lack of zeros of the partition function in certain regions of

the complex plane. In many occasions, this algorithm is known to be efficient under conditions equivalent or similar to rapid mixing conditions of Markov chains. It is an interesting direction to uncover conditions that unify both algorithmic techniques.

Cluster expansion

Cluster expansion is a technique in statistical physics to analyze systems in the low-temperature regime. On the algorithmic side, cluster expansion can be combined with both Markov chains and Barvinok's method to obtain efficient algorithms. While most other techniques only works in the high-temperature regime, cluster expansion provides unique algorithmic applications in the low-temperature regime. It is especially powerful when there are special structures that can be utilized in the underlying graphs, such as expander graphs or lattice graphs.

Overview of Talks

Geometry of Posynomials

Nima Anari, Stanford University

We generalize elements of the theory of hyperbolic and real-stable polynomials to posynomials. These are positive linear combinations of fractional monomials, where a fractional monomial is a product of fractional (not-necessarily integer) powers of variables. As our main result, we generalize a theorem of Garding, showing that homogeneous posynomials avoiding roots in a half-plane are log-concave. This quadratically improves the mixing time of natural Markov chains where the underlying distribution has a generating polynomial avoiding roots in a sector. Examples include monomer-dimer systems, non-symmetric determinantal point processes, and more. Joint work with Misha Ivkov.

Sampling proper colorings on line graphs using $(1 + o(1))\Delta$ colors

Chihao Zhang, Shanghai Jiao Tong University

We prove that the single-site Glauber dynamics for sampling proper q-colorings mixes in $O_\Delta(n \log n)$ time on line graphs with n vertices and maximum degree Δ when $q > (1+o(1))\Delta$. The main tool in our proof is the matrix trickle-down theorem developed by Abdolazimi, Liu, and Oveis Gharan. In the talk, I will demonstrate our main idea for constructing the appropriate matrix upper bounds for the local walks of the Glauber dynamics.

Uniqueness and Rapid Mixing in the Bipartite Hardcore Model

Xiaoyu Chen, Nanjing University

Counting the number of independent sets in a bipartite graph (#BIS) is a central open problem in approximate counting. Its weighted variant is known as the bipartite hardcore model. In this work, we study the bipartite hardcore model in the high-temperature regime. We show a uniqueness threshold for bipartite graphs that has almost the same form as the tree uniqueness threshold for general graphs, except with degree bounds only on one side of the bipartition. Moreover, under this uniqueness threshold, we are able to give a nearly linear time sampling algorithm and show that the standard Glauber dynamics mix in polynomial time.

MCMC in differential privacy

Jingcheng Liu, Nanjing University

Differential privacy is a formal mathematical notion of privacy. In this talk, I will tell you about a sampling problem arising from a differentially private version of graph sparsification, and how MCMC comes into play. Based on joint work with Jalaj Upadhyay and Zongrui Zou.

Something about (sampling from) log-supermodular distributions

Shuji Kijima, Shiga University

The log-supermodularity is regarded as a discrete counter part of log-concavity, particularly in the context of optimization. On the other hand, computing the partition function of a log-supermodular function is known to be #BIS-hard, which is conjectured difficult to approximate efficiently in the context of FPRAS. This talk introduces recent development by the speakers. One is about the variable transformation of a log-supermodular distribution, where we remark that the log-supermodularity is not an invariant under the symmetric difference. The other is about log $M^\natural$ convex distributions that is known to be a strict subclass of log-supermodular distributions, where we show that computing the partition function of a log $M^\natural$ convex distribution is still #BIS-hard.

[1] J. Nakashima, Y. Yamauchi, S. Kijima and M. Yamashita, Finding submodularity hidden in symmetric difference, SIAM Journal on Discrete Mathematics, 34:1 (2020), 571–585.

[2] T. Fujii and S. Kijima, Every finite distributive lattice is isomorphic to the minimizer set of an $M^\natural$-concave set function, Operations Research Letters, 49:1 (2021), 1–4.

Average local-to-global and applications

Thuy-Duong Vuong, Stanford University

In this talk, I will discuss an average version of the local to global theorem and show how it can applied to show optimal mixing of down-up walks. Unlike the standard local-to-global theorem, the average local-to-global doesn't require the precondition that all conditionals of the target distribution must have good spectral or entropic independence.

Sampling constraint satisfaction solutions in the local lemma regime

Kun He, Renmin University of China

The space of constraint satisfaction solutions is one of the most well-studied subjects in Computer Science. Given a collection of constraints defined on a set of variables, a solution to the constraint satisfaction problem is an assignment of variables such that all constraints are satisfied. A fundamental criterion for the existence of constraint satisfaction solutions is given by the Lovasz local lemma (LLL). Interpreting the space of all assignment as a probability space and the violation of each constraint as a bad event, the local lemma characterizes a regime within which a constraint satisfaction solution always exists, by the tradeoff between: (1) the chance for the occurrence of each bad event and (2) the degree of dependency between them. In Computer Science, the studies of the Lovasz local lemma are more focused on the algorithmic LLL, which is concerned with not just existence of a constraint satisfaction solution, but also how to find such a solution efficiently. In this talk, we are concerned with a problem that

we call the sampling LLL, which asks for the regimes in which a nearly uniform satisfaction solution can be generated efficiently. We will summary the new progresses and propose some open challenges.

Towards derandomising Markov Chain Monte Carlo

Chunyang Wang, Nanjing University

We present a new framework to derandomise certain Markov chain Monte Carlo (MCMC) algorithms. As in MCMC, we first reduce counting problems to sampling from a sequence of marginal distributions. For the latter task, we introduce a method called coupling towards the past that can, in logarithmic time, evaluate one or a constant number of variables from a stationary Markov chain state. Since there are at most logarithmic random choices, this leads to very simple derandomisation. We provide two applications of this framework, namely efficient deterministic approximate counting algorithms for hypergraph independent sets and hypergraph colourings, under local lemma type conditions matching, up to lower order factors, their state-of-the-art randomised counterparts.

Approximate counting for spin systems in sub-quadratic time

Jiaheng Wang, the University of Edinburgh

We present two approximate counting algorithms with $\widetilde{O}(n^{2-c}/\varepsilon^2)$ running time for some constant $c > 0$ and accuracy ε:

1. for the hard-core model when strong spatial mixing (SSM) is sufficiently fast;
2. for spin systems with SSM on planar graphs with quadratic growth, such as $\mathbb{Z}^2$.

The latter algorithm also extends to (not necessarily planar) graphs with polynomial growth, such as $\mathbb{Z}^d$, albeit with a running time of the form $\widetilde{O}\left(n^2\varepsilon^{-2}/2^{c(\log n)^{1/d}}\right)$ for some constant $c > 0$ and d being the exponent of the polynomial growth. Our technique utilizes Weitz's self-avoiding walk tree (STOC, 2006) and the recent marginal sampler of Anand and Jerrum (SIAM J. Comput., 2022).

Improved Bounds for Sampling Solutions of Random CNF Formulas

Kuan Yang, Shanghai Jiao Tong University

Let Φ be a random k-CNF formula on n variables and m clauses, where each clause is a disjunction of k literals chosen independently and uniformly. Our goal is, for most Φ, to (approximately) uniformly sample from its solution space.

Let $\alpha = m/n$ be the density. The previous best algorithm runs in time $n^{poly(k,\alpha)}$ for any $\alpha \lesssim 2^{k/300}$ [Galanis, Goldberg, Guo, and Yang, SIAM J.

Comput.'21]. In contrast, our algorithm runs in near-linear time for any $\alpha \lesssim 2^{k/3}$. This work was published in SODA'23.

An RNC Sampler for the Ising Model with External Fields

Xinyuan Zhang, Nanjing University

The Ising model is an important statistical physics model that attracts attention across various fields. A major challenge in theoretical computer science is to efficiently count and sample from the Ising model. In this talk, we will introduce the most recent progress on this problem: an RNC sampler for the Ising model with external fields. The RNC sampler is built on two key components — the spectral independence of the random cluster model and a parallel speed-up based on the "coupling with stationary" criterion.

Triangle switches: irreducibility and mixing

Catherine Greenhill, the University of New South Wales

The switch chain is a well-studied Markov chain on the set of all graphs with given degrees, with a uniform stationary distribution. I will discuss a variation of the switch chain, called the triangle switch chain, which only perform switches that change the set of triangles in the graph. We proved that each switch can be simulated by a "simulation path" consisting of a sequence of at most five triangle switches, whenever the minimum degree is at least 3. This implies that triangle switches connect the state space of graphs with a given degree sequence, under this minimum degree condition. We have also studied the mixing rate of a Metropolis implementation of the triangle switch chain, in which a graph G has stationary distribution proportional to $\lambda^{t(G)}$ for some $\lambda > 1$, where $t(G)$ is the number of triangles in G. By analysing the congestion of the simulation paths, we proved that a slight modification of this Metropolis triangle switch chain is rapidly mixing whenever the corresponding switch chain is rapidly mixing, so long as λ and the maximum degree of the chain are not too large.

Joint work with Colin Cooper and Martin Dyer.

Reversible Random Walks on Dynamic Graphs

Takeharu Shiraga, Chuo University

This talk focuses on time-inhomogeneous Markov chains. We present an upper bound on the hitting time for a random walk with time-varying transition matrices, all of which have a common stationary distribution. Our bound improves previous work on the lazy simple random walk on dynamic connected graphs with a time-invariant degree distribution. It also provides a tight bound for the lazy Metropolis walk on arbitrary dynamic connected graphs.

Sampling from convex bodies using multiscale decompositions

Piyush Srivastava, Tata institute of fundamental research

Running a random walk in a convex body $K \subseteq \mathbb{R}^n$ is a standard approach to sample approximately uniformly from the body. The requirement is that from a suitable initial distribution, the distribution of the walk comes close to the uniform distribution π on K after a number of steps polynomial in the dimension n and the aspect ratio R/r (i.e., when the body is contained in a ball of radius R and contains a ball of radius r).

Proofs of rapid mixing of such walks often require that the initial distribution from which the random walk starts should be somewhat diffuse: formally, the probability density η_0 of the initial distribution with respect to π should be at most polynomial in the dimension n: this is called a "warm start". Achieving a warm start often requires non-trivial pre-processing before starting the random walk.

This motivates proving rapid mixing from a "cold start", where the initial density η_0 with respect to π can be exponential in the dimension n. Unlike warm starts, a cold start is usually trivial to achieve. However, a random walk need not mix rapidly from a cold start: an example being the well-known "ball walk". On the other hand, Lovász and Vempala proved that the "hit-and-run" random walk mixes rapidly from a cold start. For the related coordinate hit-and-run (CHR) walk, which has been found to be promising in computational experiments, rapid mixing from a warm start was proved only recently but the question of rapid mixing from a cold start remained open.

We construct a family of random walks inspired by the classical Whitney decomposition of subsets of $\mathbb{R}^n$ into countably many axis-aligned dyadic cubes. We show that even with a cold start, the mixing times of these walks are bounded by a polynomial in n and the aspect ratio. Our main technical ingredient is an isoperimetric inequality for K for a metric that magnifies distances between points close to the boundary of K. As a corollary, we show that the coordinate hit-and-run walk also mixes rapidly both from a cold start and even from any initial point not too close to the boundary of K.

Joint work with Hariharan Narayanan (TIFR) and Amit Rajaraman (IIT Bombay).

Improved bounds for zeros of the chromatic polynomial

Viresh Patel, Queen Mary, University of London

We prove that for any graph G of maximum degree at most Δ, the zeros of its chromatic polynomial $\chi_G(z)$ (in $\mathbb{C}$) lie outside the disk of radius 5.93Δ centered at 0. This improves on the previously best known bound of approximately 6.91Δ.

We also obtain improved bounds for graphs of high girth. We prove that for every g there is a constant K_g such that for any graph G of maximum degree at most Δ and girth at least g, the zeros of its chromatic polynomial $\chi_G(z)$ lie outside the disk of radius $K_g\Delta$ centered at 0 where K_g is the solution to a

certain optimisation problem. In particular, $K_g < 5$ when $g \geq 5$ and $K_g < 4$ when $g \geq 22$ and K_g tends to approximately 3.85 as $g \to \infty$.

Key to the proof is a classical theorem of Whitney which gives a combinatorial description for the coefficients of the chromatic polynomial. This is based on joint work with Matthew Jenssen and Guus Regts.

Influences in Mixing Measures

Frederic Koehler, Stanford University

The theory of influences in product measures has profound applications in theoretical computer science, combinatorics, and discrete probability. This deep theory is intimately connected to functional inequalities and to the Fourier analysis of discrete groups. Originally, influences of functions were motivated by the study of social choice theory, wherein a Boolean function represents a voting scheme, its inputs represent the votes, and its output represents the outcome of the elections. Thus, product measures represent a scenario in which the votes of the parties are randomly and independently distributed, which is often far from the truth in real-life scenarios. We begin to develop the theory of influences for more general measures under mixing or correlation decay conditions. More specifically, we prove analogues of the KKL and Talagrand influence theorems for Markov Random Fields on bounded degree graphs with correlation decay. We show how some of the original applications of the theory of in terms of voting and coalitions extend to general measures with correlation decay. Our results thus shed light both on voting with correlated voters and on the behavior of general functions of Markov Random Fields (also called "spin-systems") with correlation decay.

Approximating the TV distance between two product distributions

Weiming Feng, the University of Edinburgh and the University of California – Berkeley

The total variation (TV) distance is a fundamental metric to measure the difference between two distributions. Recently, Bhattacharyya et. al. initiated the problem of computing the TV distance between two high-dimensional distributions. They proved that the exact computation of TV distance, even for product distributions over the Boolean domain, is #P-hard.

In this talk, I will discuss some recent progress in approximating the TV distance between two product distributions. I will introduce a randomized approximation algorithm based on the coupling technique and a deterministic algorithm based on the sparsification of distributions.

List of Participants

- Vedat Levi Alev (HUJI)
- Nima Anari (Stanford)
- Xiaoyu Chen (Nanjing)
- Weiming Feng (Edinburgh & UC Berkeley)
- Catherine Greenhill (UNSW)
- Heng Guo (Edinburgh, co-organizer)
- Kun He (Renmin University of China)
- Shuji Kijima (Shiga University)
- Frederic Koehler (Stanford)
- Jingcheng Liu (Nanjing)
- Viresh Patel (QMUL)
- Takeharu Shiraga (Chuo University)
- Piyush Srivastava (Tata institute of fundamental research)
- Thuy-Duong "June" Vuong (Stanford)
- Jiaheng Wang (Edinburgh)
- Chunyang Wang (Nanjing)
- Kuan Yang (SJTU)
- Yitong Yin (Nanjing, co-organizer)
- Chihao Zhang (SJTU)
- Xinyuan Zhang (Nanjing)

Meeting Schedule

Check-in Day: September 3rd (Sun)

- 15:00 onwards Check-in
- 19:00 - 21:00 Welcome Banquet

Day 1: September 4th (Mon)

- 09:00 - 09:30 Brief introduction
- 09:30 - 10:10 *Nima Anari*: Geometry of Posynomials
- 10:15 - 10:55 *Chihao Zhang*: Sampling proper colorings on line graphs using $(1 + o(1))\Delta$ colors
- 10:55 - 11:20 Break
- 11:20 - 12:00 *Xiaoyu Chen*: Uniqueness and Rapid Mixing in the Bipartite Hardcore Model
- 14:00 - 14:40 *Jingcheng Liu*: MCMC in differential privacy
- 14:45 - 15:25 *Shuji Kijima*: Something about (sampling from) log-supermodular distributions
- 15:25 - 16:00 Break
- 16:00 - 17:00 *Open problem session*
- Group Photo Shooting

Day 2: September 5th (Tue)

- 09:00 - 09:40 *Thuy-Duong Vuong*: Average local-to-global and applications
- 09:45 - 10:25 *Kun He*: Sampling constraint satisfaction solutions in the local lemma regime
- 10:30 - 11:00 Break
- 11:00 - 11:40 *Chunyang Wang*: Towards derandomising Markov Chain Monte Carlo
- 13:30 *Group photo shooting*
- 14:00 - 14:40 *Jiaheng Wang*: Approximate counting for spin systems in sub-quadratic time
- 14:45 - 15:25 *Kuan Yang*: Improved Bounds for Sampling Solutions of Random CNF Formulas
- 15:25 - 16:00 Break
- 16:00 - 16:40 *Xinyuan Zhang*: an RNC Sampler for the Ising Model with External Fields

Day 3: September 6th (Wed)

- 09:00 - 09:40 *Catherine Greenhill*: Triangle switches: irreducibility and mixing
- 09:45 - 10:25 *Takeharu Shiraga*: Reversible Random Walks on Dynamic Graphs
- 10:30 - 11:00 Break
- 11:00 - 11:40 *Piyush Srivastava*: Sampling from convex bodies using multiscale decompositions
- 13:30 - 18:15 Excursion
- 18:15 - 21:00 Main Banquet

Day 4: September 7th (Thu)

- 09:00 - 09:40 *Viresh Patel*: Improved bounds for zeros of the chromatic polynomial
- 09:45 - 10:25 *Frederic Koehler*: Influences in Mixing Measures
- 10:30 - 11:00 Break
- 11:00 - 11:40 *Weiming Feng*: Approximating the TV distance between two product distributions
- 13:30 Seminar close

Discussions, open problems, and future directions

Open problem: mixing time bounds for sampling from convex bodies. The problem is about improving the mixing time bounds for coordinate hit and run for sampling from convex bodies. The current dependence on the dimension n is upwards of n^7, whether from warm or cold starts. The correct exponent is likely much lower.

(Old and new) open problems related to graph colourings. A (proper) q-colouring of a simple graph is an assignment of q colours to its vertices such that no edge receives the same colour on its both endpoints. Let Δ be the maximum degree of any vertex, namely the number of its incident edges. For any constants $q \geq 3$ and $\Delta \geq 3$, it is $\#\mathbf{P}$-hard to exactly count the number of q-colourings of any graph with maximum degree Δ, even when the decision problem is trivial under the parameter setting $q \geq \Delta+1$. However, approximate counting can still be tractable, as initiated by Jerrum (1995), who first gave an fully polynomial-time randomised approximation scheme (FPRAS) for the task when $q > 2\Delta$. He therefore proposed the following well-known open problem, sometimes called the *colouring conjecture*.

Conjecture 1. *The following items are true:*

- *There is an FPRAS for the number of q-colourings in any graph with maximum degree Δ when $q \geq \Delta + 1$.*
- *The Glauber dynamics for colourings is rapid mixing when $q \geq \Delta + 2$.*

Since then, a long line of research has been devoted to the above conjecture. Some notable progresses are listed below.

- Vigoda (1999) improved the condition to $q > \frac{11}{6}\Delta$, which was later refined to $q > \left(\frac{11}{6} - \varepsilon_0\right)$ where $\varepsilon_0 = \frac{1}{84000}$ by Chen, Delcourt, Moitra, Perarnau and Postle (2019).
- For triangle-free graphs, the bound was improved to $q \geq \alpha\Delta$ where $\alpha \approx 1.7633$ is the unique positive solution of $\alpha \ln \alpha = 1$, as was shown by Chen, Galanis, Štefankovič and Vigoda (2021), and Feng, Guo, Yin and Zhang (2022) independently.
- When the graph has even larger girth, the constant can be further improved. For example, $q \geq 1.489\Delta$ suffices for graphs of girth at least 7, due to Dyer, Frieze, Hayes and Vigoda (2013). In fact, for any Δ, there exists $g = g(\Delta)$ such that $q \geq \Delta + 3$ colours suffice when the graph has girth at least g, as was shown by Chen, Liu, Mani and Moitra (2023).
- For line graphs, we can also obtain a more relaxed condition than the general case. This was first shown by Abdolazimi, Liu and Oveis Gharan (2021) that $q > \frac{10}{6}\Delta$ is enough. This was drastically improved to $q > \Delta + o(\Delta)$ by Wang, Zhang and Zhang (2023).

On the other hand, one might want to ask if the colouring conjecture asserts the *computational phase transition*. That being said, is approximate counting hard if the constant q is smaller than or equal to Δ? It makes sense to first

look at the decision problem, i.e., deciding if a proper colouring exists. This undergoes a very sharp phase transition at a threshold $q_0(\Delta) = \Delta - \sqrt{\Delta + \frac{1}{4}} + \frac{3}{2}$ such that the decision problem is tractable if and only if $q \geq q_0(\Delta)$ or $q = 2$, due to Molloy and Reed (2001). As a consequence, the approximate counting is intractable when $q < q_0(\Delta)$.

A major breakthrough of Galanis, Štefankovič and Vigoda (2015) showed, however, that the computational phase transition of the decision problem does *not* give that of the approximate counting problem. More precisely, they proved that approximate counting is **NP**-hard whenever $3 \leq q < \Delta$ and q is an even number, even when the input graph is restricted to be triangle-free. In other words, their result says the colouring conjecture, if true, lands on the point of computational phase transition, up to the parity of q and a slack $q = \Delta$.

A natural open problem is thus to settle the odd case.

Conjecture 2. *Approximate counting q-colourings of a graph with maximum degree Δ is hard whenever $3 \leq q < \Delta$.*

Though widely believed to be true, the problem is notoriously hard. The original approach is to utilise the random regular bipartite gadget (see, for example, a work by Sly (2010)), and study the so-called dominating phases of it, which immediately requires q to be an even. The same issue persists in other settings using the same gadget. For example, it was shown by Galanis, Guo and Wang (2022) that approximately counting q-colourings of a k-uniform Δ-degree hypergraph is **NP**-hard when $\Delta \geq 5 \cdot q^{k/2}$ and q is an even. However, since we do not yet pursue a tight constant in the hypergraph setting, it sounds possible to remove the parity constraint using some other approaches, assuming a slightly deteriorated condition in the form $\Delta \geq q^{(1/2+o(1))k}$.

On line graphs, although the colouring conjecture is closed to be solved, it is still unknown if the colouring conjecture gives the correct point of phase transition. To see this, note that the line graph of any d-degree graph has degree $\Delta = 2(d-1)$. The condition $q \geq (1+o(1))\Delta$ for line graphs therefore translates into $q \geq (2 + o(1))d$ for the *q-edge-colouring* on general graphs of maximum degree d. However, the only known hardness result so far is due to the decision problem: any d-degree graph is $(d + 1)$-edge-colourable by Vizing's theorem, but deciding if d colours suffice is **NP**-hard. Between $q = d$ and $q = 2d$, there is currently no evidence suggesting which would be the point of computational phase transition.

Question 3. Which $c \in [1, 2]$ marks the computational phase transition for approximately counting edge-colourings on a d-degree graph using $q = (c + o(1))d$ colours?

It is likely that an essentially new gadget will be required to tackle the above question. This is because the known approach essentially reduces from weighted independent sets (hard-core gas model), which undergoes a phase transition. However, independent sets on line graphs translate back to weighted matchings on graph (monomer-dimer model), and the latter is known to exhibit *no* phase transition.

For bipartite graphs, the algorithmic results on triangle-free graphs immediately imply the tractability when $q \geq 1.7633\Delta$. On the contrary, the best known hardness bound so far is $q \leq \Delta/(2 \ln \Delta)$, due to Galanis, Štefankovič,

Vigoda and Yang (2016). The hardness is called #**BIS**-hardness, saying the problem is no easier than approximately counting the number of independent sets in an arbitrary bipartite graph, and the latter is conjecture to be intractable by many people in the community. Filling up the gap between the algorithmic and complexity results is hence a natural problem.

Question 4. Find the correct phase transition threshold for approximately counting q-colourings in a Δ-degree bipartite graph. Does there exist any constant $c > 0$ such that the problem is intractable when $q < (c+o(1))\Delta$, assuming the intractability of #**BIS**?

We also remark that there is no parity constraint on the above hardness result. Therefore, it would be interesting to see how the approaches for general graphs and bipartite graphs can inspire each other, and answer the corresponding two questions raised here.

Open problem: the random-cluster model for $1 < q < 2$. The random-cluster model on graphs is defined as follows. Given $G = (V, E)$, it induces a distribution on subsets of edge $S \subseteq E$ (or equivalently, spanning subgraphs (V, S)):

$$\pi_{RC}(S) \propto p^{|S|}(1-p)^{|E|-|S|}q^{\kappa(S)},$$

where $p > 0$ and $q > 0$ are two parameters, and $\kappa(S)$ is the number of connected components of (V, S). This model was introduced by Fortuin and Kastelyn in the 1970s. When $q = 1$, the model is just to independently flip each edge with probability p, and on a complete graph this becomes the famous Erdös-Renyi model $G(n, p)$ of random graphs. When $q = 2$, it corresponds to the ferromagnetic Ising model, and when $q \to 0$, it corresponds to spanning trees or spanning connected subgraphs (depending on whether $p/q = o(1)$ or $p/q = \Theta(1)$). It also has various connections to Tutte polynomial, electric networks, etc.

The exact computational complexity of the random-cluster model is quite well-understood. Other than the case of $q = 1$ and the case of spanning trees, it is #P-hard to solve. On the other hand, its approximation complexity is still elusive in many cases. When $0 \leq q \leq 1$, the model can be efficiently approximated, thanks to the recent development of high dimensional expanders. When $q = 2$, the model also can be efficiently approximated, thanks to its connections to the ferromagnetic Ising model. When $q > 2$, the model can be reduced from approximating the number of independent sets in bipartite graphs, the complexity of which is still open. However, there is still essentially nothing known about the case of $1 < q < 2$. It would be surprising if the model becomes hard, but all the known algorithmic techniques do not apply.

Open problems related to parallel MCMC samping. The studies of efficient parallel algorithms for sampling and counting in general are well motivated. It is generally open and interesting to have RNC counterparts of well-known sampling and counting algorithms. A list of concrete open problems is given as follows:

- The current counting approach based on non-adaptive simulated annealing, e.g. the Chebyshev's cooling schedule used in the seminal work of

Dyer, Frieze, and Kannan for volume estimation, is sub-optimal in total work. The total work of such non-adaptive reduction from counting to sampling is in square of that of the adaptive simulated annealing of Štefankovič, Vigoda, and Vempala. A question is how much useful the polylogarithmic depth of adaptivity can be for counting via sampling. Is it possible to achieve the work efficiency of the adaptive annealing using only polylogarithmic depth of adaptivity?

- A major challenge: an RNC algorithm for sampling almost uniform (not necessarily perfect) matchings in graphs of unbounded maximum degree. It basically asks for an RNC counterpart for the Jerrum-Sinclair chain for general matchings, which is highly unknown whether it should exist.

- An even bigger challenge: an NC algorithm (with no randomness) for approximating the partition function of any near-critical graphical model with unbounded degree.

Open problems related to fine-grained complexity of counting. The study of counting complexity was initiated by Valiant (1997). For many problems (e.g. SAT and independent sets graph), the exact counting is #P complete. However, a lot of natural #P complete problems admit fully polynomial time approximation schemes (FPTAS) and fully polynomial time randomised approximation schemes (FPRAS). The approximate counting problem is extensively studied for the partition function of spin systems. Let us take the hardcore model as an example. Let $G = (V, E)$ be a graph and $\lambda > 0$ be a parameter. For every independent set $S \subseteq V$, define the weight of S by $\lambda^{|S|}$. The partition function of the hardcore model is defined by

$$Z = \sum_{\text{independent set } S \subseteq V} \lambda^{|S|}.$$

In the approximate counting problem, the input contains a graph $G = (V, E)$, a parameter $\lambda > 0$ and an error bound ϵ. We assume the graph $G = (V, E)$ has the constant maximum degree Δ and the parameter λ is also a constant. Let n denote the number of vertices in G. An FPTAS outputs a number $\widehat{Z}$ in time $\text{poly}(n, 1/\epsilon)$ such that

$$(1 - \epsilon)Z \leq \widehat{Z} \leq (1 + \epsilon)Z.$$

An FPRAS outputs a random number $\widehat{Z}$ in time $\text{poly}(n, 1/\epsilon)$ such that

$$\Pr[(1 - \epsilon)Z \leq \widehat{Z} \leq (1 + \epsilon)Z]] \geq \frac{2}{3}.$$

The approximation counting problem for the hardcore model exhibits a computational phase transition. There is a critical threshold $\lambda_c = (\Delta-1)^{\Delta-1}/(\Delta-2)^{\Delta}$ such that

- if $\lambda < \lambda_c$, then both FPTAS (Weitz 2006) and FPRAS (Anari, Liu and Oveis Gharan 2020) exists;

- if $\lambda > \lambda_c$, then approximate counting is NP-Hard. (Sly and Sun 2012)

The open problem here is to better understand the complexity of approximate counting when $\lambda < \lambda_c$.

Question 5. What is the fine-grained complexity for approximate counting when $\lambda < \lambda_c$?

For deterministic approximate counting, if $\lambda < (1-\delta)\lambda_c$, where the parameter $\delta = 1 - \frac{\lambda}{\lambda_c}$ captures the gap between λ and λ_c, Weitz's correlation decay method gives an FPTAS with running time $(\frac{n}{\epsilon})^{O_\delta(\log \Delta)}$. Another notable FPTAS is via zeros of polynomials (Barvinok 2016, Patel and Regts 2017), whose running time is in the same order as that of Weitz's algorithm. The running time of FPTAS is a huge polynomial in n. The degree of the polynomial depends on the maximum degree of the input graph. An open problem is to understand whether faster deterministic approximate counting algorithms exist.

Question 6. If $\lambda < \lambda_c$, does there exist an FPTAS with running time $f(\Delta, \lambda)(\frac{n}{\epsilon})^C$, where C is an universal constant independent from Δ and λ?

For randomised approximate counting, algorithms are all based on counting-to-sampling reduction and they have much faster running time. The seminal work of Jerrum, Valiant and Vazirani (1986) proves that for self-reducible problems, sampling and counting can be reduced to each other in polynomial time. For the hardcore model, the sampling problem is to draw random samples from the following Gibbs distribution

$$\forall \text{independent set } S, \quad \mu(S) = \frac{w(S)}{Z}.$$

A recent breakthrough by Chen, Liu and Vigoda (2021) proved that if $\lambda < \lambda_c$, then one can use Markov chain Monte Carlo algorithm to sample from μ in time $O(n \log n)$. Combining the sampling algorithm with JVV's reduction gives an FPRAS for approximate counting with running time $\widetilde{O}(\frac{n^3}{\epsilon^2})$. Another technique, simulated annealing, provides a more efficient counting-to-sampling reduction (Štefankovič, Vempala and Vigoda, 2009). Together with the above $O(n \log n)$ time sampling algorithm, it leads to an FPRAS in time $\widetilde{O}(\frac{n^2}{\epsilon^2})$. The running time is tight for this approach because any sampling algorithm (not necessarily MCMC algorithm) requires $\Omega(n)$ time to draw one random sample and the number of samples required for simulated annealing is $\Omega(\frac{n}{\epsilon^2})$. For randomised approximate counting, one natural question is to beat quadratic running time.

Question 7. For the hardcore model satisfying $\lambda < \lambda_c$,

- does there exist an FPRAS with running time $f(\Delta, \lambda) \cdot \frac{1}{\epsilon^2} \cdot n^{2-c}$ for some constant $c > 0$?
- does there exist an $\Omega(n^2)$ fine-grained lower bound for approximation counting with constant error $\epsilon > 0$.

The question is recently studied by Anand, Feng, Freifeld, Guo and Wang (2023). They prove that if $\lambda = O(\frac{1}{\Delta^k})$ for constant $k > 1.5$, then FPRAS with sub-quadratic running time exists. Note that $\lambda_c \approx \frac{e}{\Delta}$. This result does not fully answer the question above. They also consider the case when $\lambda < \lambda_c$ but the graph G satisfies certain properties. If G is a subgraph of $\mathbb{Z}^2$, then

randomised approximate counting can be solved in time $O(n^{2-c})$. If G is a subgraph of $\mathbb{Z}^d$ for $d \geq 3$, then randomised approximate counting can be solved in time $O(\frac{n^2}{f(n)})$, where $f(n)$ is greater than any polynomial of $\log n$ but smaller than any polynomial of n. So far, the above question is still open. Any progress on sub-quadratic counting in general graphs when $\lambda = O(1/\Delta)$ would be very interesting.

Open problem: sampling spectral sparsifiers. Graph sparsification has been an active area of research since its introduction, with applications such as faster algorithms in optimizations and beyond. A natural sampling version of the problem: sampling from the set of sparsifiers with weights proportional $\exp(-(\texttt{cut approximation error}))$, has recently attracted the attention in differential privacy community. Because, thanks to a standard application of the exponential mechanism, this distribution is automatically differentially private. Then, the first question is, whether the following sampling problem is computationally tractable or intractable: Given a graph G and parameters $\epsilon > 0, \beta > 0$, output H with probability proportional to

$$\Pr[H] \propto \exp\left(-\beta \cdot \max_{S \subseteq V} \min_{\alpha:|\alpha-1|<\epsilon} |E_H(S,\bar{S})/\alpha - E_G(S,\bar{S})|\right).$$

A much relaxed distribution in a recent work, that is, given a graph G, sampling H with probability proportional to $\exp(-\|G-H\|_1)$. For this distribution, a perfect sampler is found, which runs in $O(n^2)$ time. It would be nice if there is a perfect sampler that runs in $\widetilde{O}(m)$ time, i.e. in near-linear time. For approximate samplers however, it is not hard to show that a basis-exchange walk mixes rapidly for this relaxed distribution. This relaxed distribution also gives spectral approximation, but the utility is suboptimal compared to the distribution over spectral sparsifiers.

It would be good to know if the distribution that samples from all spectral sparsifiers is easy or hard to sample from. And if it is hard, is there another distribution that is easy to sample from, and is close to this distribution in some sense?