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Algorithmics for Beyond Planar Graphs

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November 28 – December 1, 2016



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Algorithmics for Beyond Planar Graphs

Organizers:

Seok-Hee Hong (University of Sydney)

Takeshi Tokuyama (Tohoku University)

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1 Summary

Many existing graph algorithms have a strong assumption that the input graph is planar, however, most graphs such as social networks or biological networks in real-world applications are *non-planar*.

The main goal of the workshop is to promote Graph Algorithm research in Asia-Pacific region, and form a research community to collaboratively solve complex problems arising in a variety of application domains such as social networks or systems biology. In particular, special emphasis on the *Beyond Planar Graphs* is addressed.

This workshop aims to identify research opportunities on Beyond Planar Graphs, focusing on the Asia-Pacific context. More specifically, we develop innovative algorithms to handle sparse non-planar graphs with specific application of large and complex network visualization:

- *k-planar graphs*: a graph is *k-planar* if it has a drawing in which no edge crosses more than *k* edges.
- *k-skew graphs*: a graph is *k-skew* if it has a drawing in which deletion of *k* edges makes the resulting graph planar.
- *k-quasi-planar graphs*: a graph is *k-quasi-planar* if it has a drawing in which no *k* edges mutually cross each other.

This workshop aims to bring world-renowned researchers on Graph Algorithm, Graph Drawing, Computational Geometry, Graph Theory, and Combinatorial Optimization, and collaboratively develop innovative theory and algorithms for sparse non-planar topological graphs with specific applications of large and complex network visualization.

More specifically, we have the following aims:

- **Structural properties**: We aim to characterize classes of sparse non-planar topological graphs. We prove important structural properties of such graphs.
- **Testing algorithm**: We want to determine the complexity of the problem of testing whether a given graph satisfies such topological constraints is NP-hard. If not, we design efficient (polynomial time) algorithms for testing such properties.
- **Drawing algorithm**: We aim to design polynomial time algorithms to construct a straight-line drawing of an embedding that satisfies topological constraints.

Our specific objectives are:

- Identify research opportunities on Beyond Planar Graphs, focusing on the Asia-Pacific context.
- Form a broader research community with cross-disciplinary collaboration, between Computer Science (Graph Algorithm, Theoretical Computer Science, Combinatorial Optimization) as well as Mathematics (Graph Theory, Combinatorics).
- Foster greater exchange between Graph Drawing community, Computational Geometry community and Graph Theory community, and to draw more researchers in the Asia-Pacific region to enter this rapidly growing area of research.
- Assist emerging researchers to find linkages to international researchers and competitive research grants and funding.

2 Program

- Nov 27 Sunday Evening
 - Reception
- Nov 28 Monday Morning
 - Introduction
 - Invited Talk 1: Peter Eades
 - Invited Talk 2: Fabrizio Montecchiani
- Nov 28 Monday Afternoon
 - Invited Talk 3: Ignaz Rutter
 - Invited Talk 4: Martin Nöllenburg
 - Invited Talk 5: Yoshio Okamoto
 - Invited Talk 6: Luca Grilli
 - Invited Talk 7: Seok-Hee Hong
 - Open Problem Session
- Nov 29 Tuesday Morning
 - Open Problem Session
 - Group Discussion I
- Nov 29 Tuesday Afternoon
 - Group Discussion II
 - Group Report I
- Nov 30 Wednesday Morning
 - Group Discussion III
 - Group Report II

- Nov 30 Wednesday Afternoon
 - Afternoon: Excursion to Kamakura
- Dec 1 Thursday Morning
 - Group Discussion IV
 - Group Report III
 - Planning and Wrap up

3 Invited Talks

Straight-line Drawings for Nearly Planar Topological Graphs

Peter Eades, University of Sydney, Australia

We discuss three classes of topological graphs that are “nearly” planar in some sense. In particular, we investigate whether graphs in these classes admit straight-line drawings.

1. A **1-skew graph** $G = (V, E)$ is a graph with an edge (s, t) such that $G' = (V, E - \{(s, t)\})$ is planar. Suppose that G has a straight-line drawing D in which the edge (s, t) crosses edges e_1, e_2, \dots, e_k . It is simple to observe that for each i , one endpoint of e_i is left of (s, t) and the other is right of (s, t) . This simple observation leads to an elegant theorem characterizing those topological graphs on the sphere that have a straight-line drawing in the plane [6]. To prove this theorem, we present a characterization of maximal 1-skew topological graphs that have a straight-line drawing in the plane. The proofs of these theorems are algorithms. Further, we describe an interesting proof technique using a variant of Tutte’s barycentre algorithm.
2. A **1-plane graph** is a topological graph in which each edge has at most one crossing. Thomassen [21] characterized those 1-plane graphs that admit a straight-line drawing in terms of two forbidden subgraphs. We describe a linear-time algorithm which tests for these forbidden subgraphs, and constructs a straight-line drawing if the forbidden subgraphs are absent [14]. In particular, we describe an augmentation technique that adds edges to increase connectivity.
3. A **RAC (right-angle crossing) graph** is a topological graph that admits a straight-line drawing in which each edge crossing forms a right angle. Research on RAC graphs is motivated by human experiments that have shown that right-angled edge crossings do not inhibit human understanding of diagrams. In particular, it has been shown that a RAC graph with n vertices has at most $4n - 10$ edges [5]; further, if it has exactly $n - 10$ edges then it is 1-planar [7].

An interesting open problem is to characterize those 1-planar topological graphs that have a RAC drawing. Dehkordi [3] conjectures that six forbidden subgraphs characterize those 1-plane graphs that have RAC drawings.

Geometric Representations of 1-Planar Graphs

Fabrizio Montecchiani, University of Perugia, Italy

A graph is *1-planar* if it can be drawn in the plane such that each edge is crossed at most once. Among the various families of beyond planar graphs recently investigated in the literature, the 1-planar graphs are among the most popular ones. We first briefly recall the main results concerning both the density of 1-planar graphs and the complexity of recognizing 1-planar graphs.

After this introduction, we review the main results concerning *geometric representations* of 1-planar graphs. Namely, we first describe some results and proof techniques for straight-line drawings and 1-bend drawings with right-angle crossings and with few edge *slopes*.

We then turn our attention to *visibility representations* of 1-planar graphs. In particular, we describe recent results and proof techniques concerning bar 1-visibility representations, rectangle visibility representations, and ortho-polygon visibility representations.

My open problems include:

- The 1-planar slope number of 1-planar graphs with maximum vertex degree Δ is the minimum number of edge slopes that is sufficient to compute a 1-planar straight-line drawing of any 1-planar graph with maximum vertex degree Δ . We ask if this number can be bounded by a function that depends only on Δ and does not depend on the number of vertices of the graph.
- An edge partition of a 1-plane graph G (i.e., of a graph with a given 1-planar embedding) is a coloring with two colors of the edges of G , say red and blue, such that both the graph induced by the red edges and the graph induced by the blue edges are plane. We ask whether every 3-connected 1-plane graph has an edge partition such that the red graph has vertex degree at most c , for some constant c , and treewidth at most two.

Simultaneous Embeddings and Stream Planarity

Ignaz Rutter, TU Eindhoven, The Netherlands

Motivated by the problem of visualizing sequences of graphs representing different time steps of dynamic graphs, we survey results and techniques for simultaneous embeddability and testing the planarity of a stream of edges, where only the edges contained in a sliding window of fixed size are simultaneously visible.

For *simultaneous embeddings* we focus on the main variants of *straight-line geometric embeddings* (SGE) and topological drawings where shared edges have to be represented by the same curve. The latter are also called *simultaneous embeddings with fixed edges*, or SEFE for short. While the main body of the literature on SGE focuses on determining graph classes that do always or do not always admit an SGE, and few algorithmic results are known, the literature for SEFE contains algorithms for a wide range of instances, though the complexity of the general case is still open.

We give an overview of the techniques used to approach special cases of the SEFE problem. Our coverage of the SEFE problem ends with a discussion of the geometric realizability problem, which asks for a realization of a SEFE drawing with polylines that use few bends. One of our main open problems is to reduce the gap between the

best known lower bound of three bends per edge and the best known upper bound of 6 bends per edge.

Finally, we introduce *stream planarity* as an attempt to overcome the hardness of SEFE for three or more input graphs, which limits its application to very short sequences of graphs, only. In a stream graph the edges are represented in the form of a stream and a sliding window of fixed size is used to determine the subgraph for each time step. In this way, the graph sequences can be very long, but the amount of change between any two time steps is very limited, in the sense that only one edge is added and one disappears. Here we survey the main results and relate them other recently studied graph embedding problems.

Geometric Graph Layouts of Beyond Planar Graphs

Martin Nöllenburg, TU Vienna, Austria

Crossings represent a major nuisance for reading and understanding drawings and visualizations of graphs. Non-planar graphs, however, cannot be drawn in the plane without crossings. In this talk I survey different approaches and techniques for improving the aesthetics of geometric graph layouts of non-planar graphs.

The first part of the talk covers three different approaches to improve the visual appearance of crossings:

- In a *slanted orthogonal drawing*, a graph with maximum degree 4 is drawn such that all edges are sequences of horizontal, diagonal, and vertical line segments with bend angles of 135° . Moreover, crossings are only permitted on diagonal segments, which makes them visually distinguishable from vertices, whose incident edges connect only vertically and horizontally. Moreover, all crossings have 90° crossing angles. Different optimization problems have been studied in the literature.
- The second approach uses *edge casing*, i.e., for each crossing one edge is above the other edge and the lower of the two edges is interrupted by a small gap to indicate this. Different optimization problems for the sequences and the geometry of edge casings in the stacking and weaving models have been proposed and studied in the literature.
- The third approach uses *bundled crossings* to reduce the number of perceived crossings. Here a bundled crossing is defined by the mutual crossings of all pairs of edges from two disjoint edge sets E_1 and E_2 such that all these crossings can be enclosed by a (small) pseudodisk. The complexity and algorithms for minimizing the number of bundled crossings have been studied in the literature.

In the second part of the talk two techniques are presented that make crossings disappear visually by not showing them explicitly:

- In *confluent drawings* of graphs, the edges are represented as smooth paths between their respective endpoints that are composed of sequences of non-crossing curves that pass through a set of junctions, similarly to a schematic layout for a system of train tracks. In each junction all incoming curves on one side connect to all incoming curves on the other side and vice versa. Several results on confluent drawings are known in the literature, e.g., classes of graphs that have or do not have confluent drawings.

- Finally, *partial edge drawings* (PEDs) are straight-line graph drawings in which the middle part of each edge is erased. Using the closure principle of Gestalt theory, humans may still see the link between two vertices even if some part of it is missing. This can strongly reduce the number of crossings shown in the layout. The literature provides several results on such PEDs, e.g., graphs that admit or do not admit a PED, as well as algorithms for optimizing the remaining edge stubs.

Angular Resolution — Around Vertices and Crossings

Yoshio Okamoto, UEC, Japan

Angular resolution is one of the traditional criteria for aesthetic graph drawing. While the classical work was only concerned with *vertex angular resolution*, which is defined as the minimum angle formed by two edges incident to a vertex, more recent work has started to look at *crossing angular resolution*, which is defined as the minimum angle formed by two crossing edges. The *total angular resolution* is then defined as the minimum of the vertex angular resolution and the crossing angular resolution.

In this talk, we look at traditional and recent results on angular resolutions, and identify some open problems that relate other well-known concepts in graph drawing.

Testing Fan-planarity and Maximal Outer-fan-planarity

Luca Grilli, University of Perugia, Italy

A *fan-planar drawing* is a simple drawing that can be formally defined in terms of two forbidden crossing patterns: (i) an edge crossed by two independent edges, and (ii) an edge e crossed by two incident edges having their common endpoint on different sides of e [17]. A *fan-planar graph* is a graph that admits a fan-planar drawing.

An *outer-fan-planar graph* is a graph that admits an *outer-fan-planar drawing*, which is a fan-planar drawing with all vertices along the external boundary. A *maximal outer-fan-planar graph* is an outer-fan-planar graph such that the addition of any edge destroys its outer-fan-planarity.

A *2-layer fan-planar graph* is a graph that admits a *2-layer fan-planar drawing*, i.e. a fan-planar drawing where vertices are placed on two distinct horizontal lines and edges are vertically monotone lines.

In this talk, we discuss the main results concerning fan-planar graphs.

- **Density:** Kaufmann and Ueckerdt [17] proved that every n -vertex fan-planar graph, without loops and multiple edges, has at most $5n - 10$ edges, which is a tight bound for $n \geq 20$. Binucci et al. [2] showed that n -vertex outer-fan-planar graphs have at most $3n - 5$ edges, and that this bound is tight for $n \geq 5$.
- **Hardness of fan-planarity:** Binucci et al. [2] have shown that testing fan-planarity in the variable embedding setting is NP-complete; the NP-hardness follows by a reduction from the 1-planarity testing problem, which is NP-complete in the variable embedding setting [12, 18]. We show that testing fan-planarity remains NP-complete even in the fixed rotation system setting [1], using a reduction from the 3-partition problem.

- **Testing maximal outer-fan-planarity:** We present a linear-time algorithm to test whether a given graph G is maximal outer-fan-planar [1]. In the affirmative case, the algorithm provides an outer-fan-planar embedding of G . We first consider the 3-connected case and give a linear-time testing algorithm that is able to produce all the outer-fan-planar embeddings, in case of a positive answer; the number of outer-fan-planar embeddings is at most twelve. Finally, we obtain a linear-time testing algorithm for 2-connected graphs, using the decomposition of the input graph into its 3-connected components, called an SPQR-tree data structure.

Testing Outer-1-planarity and Full Outer-2-planarity

Seok-Hee Hong, University of Sydney, Australia

A graph is *1-planar* if it can be embedded in the plane with at most one crossing per edge. It is known that the problem of testing 1-planarity of a graph is NP-complete [12, 18].

A graph is *outer- k -planar* for an integer $k \geq 0$, if it admits an *outer- k -planar embedding*, that is, every vertex is on the outer face and no edge has more than k crossings. A graph is *full-outer- k -planar*, if it admits a *full-outer- k -planar embedding*, that is, an outer- k -planar embedding such that no crossing appears along the outer face.

In this talk, we first present a linear time algorithm to test whether a given graph is outer-1-planar [13]. The algorithm can be used to produce an outer-1-planar embedding in linear time if it exists.

Next, we present linear-time algorithms for testing full-outer-2-planarity of a graph G , where G is connected, biconnected or triconnected [15]. The algorithm also produces a full-outer-2-planar embedding of a graph, if it exists. In particular, we prove several structural properties of triconnected outer-2-planar graphs and full-outer-2-planar graphs, and show that every triconnected full-outer-2-planar graph has a constant number of full-outer-2-planar embeddings.

4 Working Group 1: Confluent Thickness

Confluent drawings [4] are an unconventional drawing style to draw non-planar graphs in a planar way. They consist of a set of graph vertices (represented as points) and a set of junctions, connected by a crossing-free set of smooth curves. In each junction two bundles of curves enter a single point from opposite sides, such that they share a common tangent in the junction point. Now two vertices u and v are connected by an edge in the underlying graph if and only if there is a locally monotone path (a smooth path without sharp bends) from u to v via a sequence of junctions. One can imagine the layout as a set of train tracks and two vertices are connected by an edge if and only if a train can drive from one to the other without changing its travel direction. In such a layout, each junction realizes all pairwise links between the two incident bundles without showing any explicit crossings, see Fig. 1 for an example.

Several results are known on confluent drawings, e.g. [4, 9, 8, 10, 16]. For some graph classes it is known that they admit a confluent drawing, e.g., interval graphs, complements of trees, or cographs. On the other hand, there are graphs that do not have a confluent drawing, e.g., the Petersen graph, the 4D hypercube, or certain subdivisions

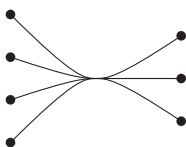


Figure 1: A confluent drawing of $K_{4,3}$.

of non-planar graphs. The complexity of deciding whether a graph admits a confluent drawing is open.

The *thickness* of a graph G is the minimum number k such that G can be decomposed into k planar subgraphs whose union is G . If each of these subgraphs is colored in a distinct color, G can be drawn with colored edges such that no two edges of the same color cross.

Our goal in the working group was to study confluent drawings of *confluent thickness* 2, i.e., the arcs of a confluent drawing are colored red and blue such that each monochromatic sub-layout is a planar confluent drawing, but arcs of different color may cross. We call such a drawing *2-confluent*. We may optionally allow some arcs to have both colors. Obviously, these definitions may be generalized to confluent thickness $k > 2$.

During our working group sessions in Shonan, we obtained the following results and observations:

4.1 Bipartite layouts on parallel lines

A common way to draw bipartite graphs is to put the two partitions of vertices on two vertical lines with all edges inside the strip between the lines. In this way a full bipartite graph can be drawn using confluence by merging all edges from either side into a single edge (recall Fig. 1). We found a family of graphs that can be drawn in this manner using two colors. This family is the bipartite complement of a set of disjoint bipartite cliques. Such a graph can be drawn as in Figure 2 (right). Intuitively, there are *bridges* that connect from the left to the right. These bridges are positioned between the cliques of the complementary graph. Each left-side part of a clique connects to bridges above it in red and to bridges below it in blue. Each right-side part of a clique connects to bridges above it in blue and bridges below it in red. As a consequence from a vertex one can reach all other vertices in red or blue except those from the same clique as this would require going up and down from a bridge in two different colors (black lines are both red and blue).

However, we could construct a bipartite graph that does not have bipartite layout on parallel lines. In particular a set of five complete bipartite graphs that have a unique edge in common does not have such a representation. The proof is based on the fact that the edge shared by the cliques has to be represented with at least five smooth curves, so called *branches*, as in Figure 3. At least three of the five branches have the same color, say red. The red cliques that lie either above or below the shared edge need to connect to the branches. They can not connect to them without creating a red-red crossing.

4.2 Mycielski construction

We also studied whether an *embedded quasi-planar graph*, i.e., a graph that can be represented in the plane without three mutually crossing edges, can be drawn confluent

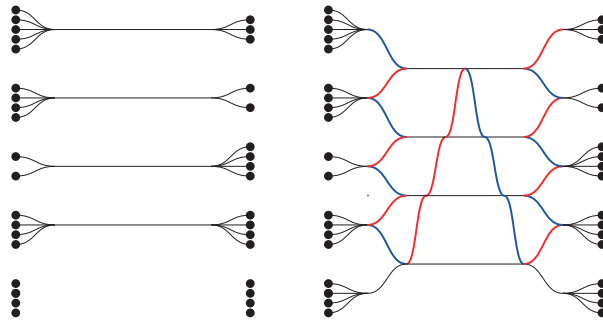


Figure 2: A confluent drawing of a set of disjoint bipartite cliques (left) and its complement drawn confluent with two colors.

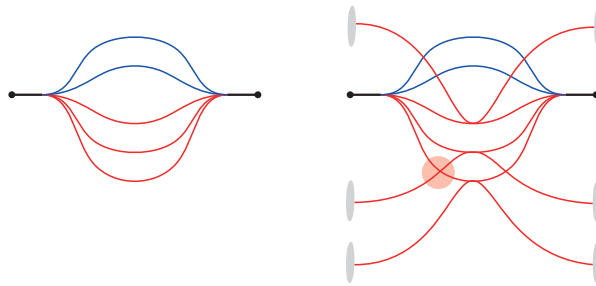


Figure 3: (Left) an edge shared by five cliques have to be represented by five branches; (Right) three cliques connecting to three red branches create red-red crossing.

using a constant number of colors. We showed that there exists no constant c such that every graph of this family is c -confluent. We did this by considering the iterated Mycielskian graphs, a family of recursively defined graphs with the property that the Mycielskian graph M_i requires i colors for a valid vertex coloring. Since these graphs are triangle-free, we can use them as the dual of our quasi-planar graphs, where every edge of M_i corresponds to a line segment (edge between two vertices) in the quasi-planar graph and two edges of the quasi-planar graph intersect each other when their corresponding vertices in M_i share an edge. We showed how to construct the primal graph of M_i , starting from the primal graph of M_{i-1} . Finally, since the family of Mycielskian graphs is infinite, we showed that the primal graph of M_i requires i colors and since no vertices in the primal graph share an edge, confluency does not change this.

4.3 Computational hardness

We consider the following problem, where we require that the given embedding must be preserved:

Input: A graph G and its embedding.

Question: Find the minimum number k such that G and its embedding have a k -confluent drawing.

We show that this problem is NP-hard by reducing from the vertex coloring problem. The vertex coloring problem is known to be NP-hard even for planar graphs. Let a planar graph G' be an instance of the vertex coloring problem. To construct G , we replace each vertex in G' by a curve (here, drawn as a polyline). Then we give an embedding such that each pair of edges in G share a bundle if and only if the corresponding vertices in G' are adjacent (see Figure 4, where the black boxes indicate a double crossing of the curves). By construction, such a pair of edges must have different colors. Trivially, there exists a k -coloring for G' if and only if G and its embedding have a k -confluent drawing. Note that, the fact G' is planar implies that always there exists such an embedding.

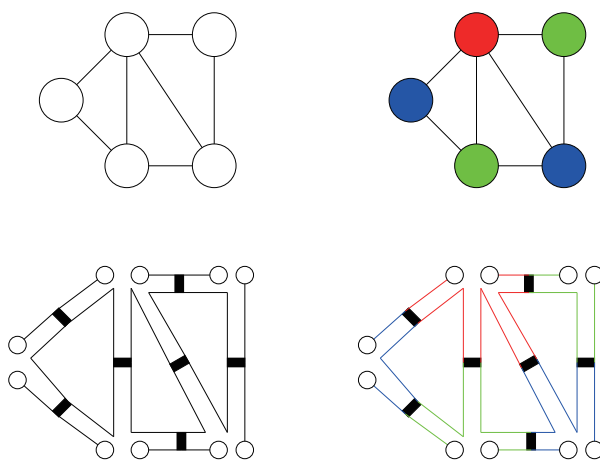


Figure 4: Reducing from the vertex coloring problem.

We do not yet know if a graph with confluent thickness larger than two exists (assuming no fixed embedding is required). We believe there should exist one, but so far we have failed to find an example. This is one of the main problems left for the future.

5 Working Group 2: k -Blip Graphs

5.1 Asymmetric crossing numbers

A *crossing* between two edges in drawing of a graph is a symmetric relation, and one typically seek a drawing that minimizes the number of crossings. On day one of the workshop, Martin Nöllenburg surveyed optimization results by Eppstein et al. [11] on the so-called *cased drawings*, which order the edges of a crossing and interrupts the lower edge in an appropriate neighborhood of the crossing, in order to improve readability. Eppstein et al. [11] solved several optimization problems for such *antisymmetric* crossings, however, they did not consider the properties near planarity. Recall that a drawing of a graph is called k -*plane* if every edge crosses at most k other edges; and a graph is called k -*planar* if it admits a k -plane drawing.

The following definition, suggested at the Open Problem session of the workshop, combines cased drawings and k -planarity. Let $k \in \mathbb{N}$.

- A drawing of a graph is called a k -*blip drawing* if each edge crossing can be

assigned to one of the two crossing edges such that no more than k crossings are assigned to each edge in E .

- A graph is called a k -blip graph if it admits a k -blip drawing.

One of the research groups at the workshop explored the properties of k -blip graphs. The following three sections summarize the results obtained in Shonan or shortly after the workshop.

5.2 Basic properties

Using Hall's theorem on bipartite matchings, we have shown that every $(2k)$ -planar graph is k -blip. However, the converse is false. Furthermore, for every $k \in \mathbb{N}$, we have constructed a graph G_k that is k -blip but not 1-planar. If every edge crosses at most $2k$ others, then the crossings can be distributed among the edges to minimize the maximum load on an edge.

Further, a k -blip graph admits a drawing in which there are at most $|E'|$ crossing between any subset of $E' \subseteq E$ of edges. This implies a relation with quasi-planarity. Recall that a graph G is q -quasiplanar, for $q \in \mathbb{N}$, if it admits a drawing in which there is no subset of q pairwise crossing edges, hence no more than $\binom{q}{2} = q(q-1)/2$ crossings. In a k -blip drawing, there are at most kq crossings among q edges. Consequently, a k -blip graph is $(2k+1)$ -quasiplanar.

$$(2k)\text{-PLANAR} \subset k\text{-BLIP} \subset (2k+1)\text{-QUASIPLANAR}$$

5.3 Density

Every k -blip graph $G = (V, E)$ admits a drawing with at most $k|E|$ crossings (by double counting the number of crossing-edge assignments). This already implies that k -blip graphs are sparse for any constant k . A k -blip n -vertex graph has at most $O(\sqrt{k} \cdot n)$ edges. Indeed, the crossing number of a graph G with n vertices and m edges is bounded by $\text{cr}(G) \geq \frac{1024}{31827} \cdot m^3/n^2$ when $m \geq \frac{103}{6}n$ [19]. Combined with the bound $\text{cr}(G) \leq km$, we obtain

$$\frac{1024}{31827} \cdot \frac{m^3}{n^2} \leq \text{cr}(G) \leq km \quad \implies \quad m \leq \max(5.58\sqrt{k}, 17.17) \cdot n.$$

We have tried to improve the upper bounds for small values of k , in particular for $k = 1$. Pach et al. [19] proved that a graph G with $n \geq 3$ vertices satisfies $\text{cr}(G) \geq \frac{7}{3}m - \frac{25}{3}(n-2)$. Combined with the bound $\text{cr}(G) \leq km$, we have

$$m \leq \frac{25(n-2)}{7-3k}.$$

For $k = 1$, this gives $m \leq 6.25n - 0.5$ for $k = 1$. This is the current best bound for the density of 1-blip graphs.

We believe that these upper bounds can be improved. As noted above, every 2-planar graph is 1-blip. A 2-planar n -vertex graph has at most $5n - 10$ edges, and this bound is the best possible [20]. Even though the class of 1-blip graphs contains all 2-planar graphs, we have not found any 1-blip graph with more than $5n - 10$ edges. Nevertheless, we have found several constructions for 1-blip n -vertex graphs with $5n - \Theta(1)$ edges; see Figure 5.

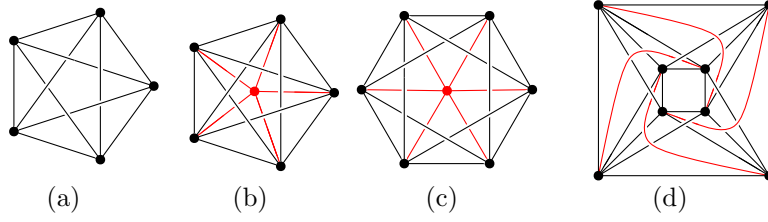


Figure 5: Four patterns that produce for 1-blip graphs with n vertices and $5n - \Theta(1)$ edges.

A 2-planar graph with n vertices $5n - 10$ edges is also 1-blip. Pach and Tóth [20] construct such a graph by starting with a plane graph with pentagonal faces (e.g., using nested copies of an icosahedron), and then add all five diagonals in each pentagonal face; see Fig. 5(a). We can modify this construction by inserting new vertex in any pentagon, and connect it to the 5 vertices of the pentagon; Fig. 5(b). Every new edge crosses exactly one diagonal of the pentagon, so the new crossings can be charged to the new edges. Since the new vertices have degree 5, the bound $m \leq 5n - 10$ prevails.

A similar construction is based on hexagonal faces; see Fig. 5(c). Start with a *fullerene*, a 3-regular, plane graph G_0 with n_0 vertices, 12 pentagon faces, and $n_0/2 - 10$ hexagon faces (including the external face). Add diagonals in each face to connect a vertex to their 2nd neighbors (the graph is 2-planar so far); finally insert a new vertex in each face of G_0 , and connect them to all vertices of that face. We obtain a 1-blip graph G . The number of vertices is $n = n_0 + 12 + (n_0/2 - 10) = \frac{3}{2}n_0 + 2$, and the number of edges is $\frac{3}{2}n_0 + 10 \cdot 12 + 12 \cdot (n_0/2 - 10) = \frac{15}{2}n_0 = 5n - 10$.

A third construction, is based on a sequence of nested squares; Fig. 5(d) shows how to add 16 edges between two consecutive squares such that the 16 crossings are assigned to distinct edges. We can add two diagonals in the external face and the innermost square. Using s squares, we have $n = 4s$, and $m = 4s + 16(s - 1) + 2 \cdot 2 = 20s - 12 = 5n - 12$.

5.4 Recognition

If we are given a drawing of a graph, the crossings can be efficiently distributed among the edges to minimize the maximum load on an edge, as noted by Eppstein et al. [11]. This is due to the fact that directing the edges of the crossing graph of Γ such that the maximum indegree is minimized, can be solved in time quadratic in the number of edges of the crossing graph [22].

If no drawing is given, we face the recognition problem for 1-blip graphs: Given graph $G = (V, E)$, decide whether G is a 1-blip graph. We conjecture that the problem is NP-hard, perhaps by a reduction from 1-planarity, which is known to be NP-hard [12, 18]. The key tool for a possible reduction would be a “blocker gadget,” which is a 1-blip graph that has only one 1-blip drawing (up to isometries). We have considered complete graphs and complete bipartite graphs.

A complete graph on n vertices, K_n , is 1-blip if and only if $n \leq 8$. Figure 6 shows a 1-blip drawing of K_8 . The crossing number of K_n , for $n \geq 10$, is larger than the number of edges $\binom{n}{2}$, and so it is not 1-blip. For $n = 9$, we have $\text{cr}(K_9) = 36$ and K_9 has $\binom{9}{2} = 36$. Consequently, in any 1-blip drawing of K_9 must have $\text{rmcr}(K_9)$ edge crossings and every edge is charged for precisely one crossing. A short argument

shows, however, that no such drawing exists, consequently K_9 is not a 1-blip graph.

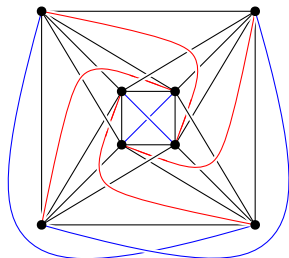


Figure 6: A 1-blip drawing of K_8 .

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