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Algorithmic Randomness and Complexity

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National Institute of Informatics
2-1-2 Hitotsubashi, Chiyoda-Ku, Tokyo, Japan

Algorithmic Randomness and Complexity

Organizers:
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Information and randomness are fundamental notions of computer science. Their practical applications are ever increasing. For instance, a recent volume of *Communications of the ACM* describes an application of (approximations of) Kolmogorov complexity to voice recognition. Kolmogorov complexity can be used to measure information distance in a natural and context-free way, and can be used in areas as diverse as computational biology, music evolution, data mining, and at least 20 other areas. Classical information theory and probability provide formalizations, but do not allow us to speak of the information content of a specific string or to say that a particular infinite binary sequence is random. These notions were formalized in the 1960s with the introduction of Kolmogorov complexity and Martin-Löf randomness. The first measures the complexity (or information content) of finite binary strings, while the second captures the intuition that a random infinite binary sequence has no "distinguishing features". There are also characterizations of Martin-Löf randomness that capture the alternative intuitions that a random sequence is unpredictable and incompressible, the latter connecting Martin-Löf randomness to Kolmogorov complexity.

In recent years the people working in computability theory have produced a surge that resulted in rapid progress in our understanding of even the most basic notions, and in the solution of old open questions. This progress has changed and is still changing the landscape and opened up new avenues of research. An evidence of this activity has been the publication of two new books in the area, and the new edition of an already classical one.

1. *Algorithmic Randomness and Complexity*, R. Downey and D. Hirschfeldt, Theory and Applications of Computability, Springer, 2010.

2. *Computability and Randomness,* A. Nies, Oxford University Press, 2009.

3. *An Introduction to Kolmogorov Complexity and Its Applications,* M. Li and P. Vitanyi. Third Edition, Springer Verlag, 2008.

Additionally, two of the proposers have been invited to the *International Congress of Mathematicians* to speak on algorithmic randomness, and a number of special issues of Computer Science Journals have been devoted to this area.

1. R. Downey, Algorithmic randomness and computability, *Proceedings of the 2006 International Congress of Mathematicians,* Vol. 2, *European Mathematical Society*, 2006, 1–26.

2. A. Nies, Interactions of computability and randomness. S. Ragunathan (ed.), R. Bhatia (ed.), *Proceedings of the 2010 International Congress of Mathematicians,* Vol. 2, 2010, 30-57.

3. R. Downey (ed.), special issue of *Theory of Computing Systems* for CCR 2011, Theory of Computing Systems, Vol. 52, Issue 1, 2013,

4. E. Mayordomo and W. Merkle (eds.), special issue of *Theory of Computing Systems* for CCR 2012, to appear.

Research on the notions of information and randomness has drawn on methods and ideas from computability theory and computational complexity, as well as from core mathematical subjects like measure theory and information theory. A major goal of the seminar is to bring together researchers to discuss the current open problems.

Our meeting was structured as follows: We had round lectures in the morning which involved thematic days; the talks concentrating on open questions together with an introductory lecture. Each afternoon (except the excursion) we met for an hour to review the morning lectures and discuss the open questions as a whole group. At 3.00 we split into several smaller groups to work on problems mostly related to ones suggested this week.

Free movement between the small groups was encouraged. We give the abstracts of the relevant lectures below and after that discuss new work and collaborations generated by the meeting.

Overview of Talks

Randomness relative to an enumeration oracle

Joseph S. Miller, UW-Madison

We initiate the study of algorithmic randomness relative to an enumeration oracle. Most of the notions in algorithmic randomness, and all of those most closely related to Martin-Löf randomness, can be expressed in terms of c.e. sets. So when we relativize these notions to an oracle X, we are usually interested in the X-c.e. sets, rather than the X-computable sets. Generalizing from Turing degrees to enumeration degrees should be straightforward: to relativize a randomness notion to the enumeration degree of a set A we simply replace "A-c.e." with "c.e. in every enumeration of A" (i.e., enumeration reducible to A). The resulting definitions are easily seen to be unchanged for the total degrees (the range of the Turing degrees under their natural embedding into the enumeration degrees). For nontotal degrees, we find that some familiar theorems are preserved, perhaps with more subtle proofs, while other theorems may fail. For example, there need not be a universal Martin-Löf test relative to the enumeration degree of A, and there are continuum many enumeration degrees that are low for randomness. These and other facts follow from studying specific classes of enumeration degrees; unsurprisingly, properties of an enumeration degree are reflected in the behavior of randomness notions relative to that degree.

This work is joint with Mariya Soskova.

Diamond embeddings and minimal pairs in the r.e. truth table degrees

Keng Meng Ng, NTU

We present some recent and old results on minimal pairs in the r.e. truth table degrees. Jockusch and Mohrherr showed that the diamond lattice $\{a, b, 0, 1\}$ can be embedded in the r.e. truth table degrees preserving the top and bottom elements. We discuss related results about the possible degrees of a and b. We also discuss how to use the determinacy of finite games to construct minimal pairs in the r.e. truth table degrees.

Open problems on reducibilities in Algorithmic randomness

George Barmpalias, Victoria U, Wellington

This talk is based on my recent BSL paper on the topic. I will present the state-of-the art of measures of relative randomness as reducibilities measuring randomness and related concepts.

Techniques in randomness and ergodic theory

Johanna Franklin, Hofstra University

Many different results concerning the relationship between algorithmic randomness and effective ergodic theory have been proven in recent years. In this talk, I 'll survey the approaches that have been taken and the results from classical ergodic theory that have been used in the proofs of these results, focusing on my own work with Towsner and the method of cutting and stacking.

Effective Kolmogorov-Sinai and Topological Entropy

Satyadev Nandakumar, IIT Kanpur

Symbolic dynamics is a useful approach for generalizing theorems in a symbolic setting to general settings of the dynamics of measure-preserving maps on metric spaces. We give an exposition on two important notions of complexity in symbolic dynamics, viz. the notion of Kolmogorov-Sinai entropy and topological entropy, and the connection between the two via the Variational Principle. Moreover these notions of entropy can also be a tool for the classification of dynamical systems if two measure-preserving dynamical systems have a measure-preserving isomorphism between them, then they have the same Kolmogorov-Sinai entropy. We give an overview of effectivizations of these notion, by Brudno, White, and by Galotolo, Hoyrup and Rojas, in terms of the complexity of orbits of Martin-Löf random objects in effective dynamical systems. We also explain the completeness of Kolmogorov-Sinai entropy as an invariant, in the context of Bernoulli systems that is, two Bernoulli Systems are isomorphic if and only if they have the same KS entropy. We explain a recent attempt at the effectivization of this result, and suggest currently open areas of research.

Effective Multifractals and Renyi-Entropy

Jan Reimann, Pennsylvania State University

Multifractal measures play an important role in the study of point processes and strange attractors. A central component of the theory is the multifractal formalism, which connects local properties of a measure (pointwise dimensions) with its global properties (average scaling behavior).

I will introduce a new, effective multifractal spectrum, where one replaces pointwise dimension by asymptotic compression ratio. It turns out that the underlying measure can be seen as a universal object for the multifractal analysis of computable measures. The multifractal spectrum of a computable measure can be expressed as a " deficiency of multifractality " spectrum with respect to the universal measure. This in turn allows for developing a quantitative theory of dimension estimators based on Kolmogorov complexity. I will discuss some applications to seismological dynamics.

The Minimum Circuit Size Problem, and Kolmogorov Complexity

Eric Allender, Rutgers University

The Minimum Circuit Size Problem (MCSP) is a well-known example of a set in NP that is not known (or widely believed) to be NP-complete, although it appears to be intractable. It has recently been shown to be hard for the complexity class SZK (Statistical Zero Knowledge).

MCSP is closely related to the set R_{KT} — the set of KT-random strings (where KT is a type of time-bounded Kolmogorov complexity). The MCSP-hardness result also holds for a " promise problem " version of R_{KT}: The yes instances consisting of all x such that $KT(x) > |x| - 1$. The no instances consisting of all x such that $KT(x) < |x|/2$.

Investigating the " promise problem " version of the sets reducible to R_K may lead to improved connections between complexity theory and the study of K-randomness (such as attempts to characterize classes such as NEXP and BPP in terms of efficient reductions to R_K.

Sets efficiently reducible to Kolmogorov randomness

Akitoshi Kawamura, The University of Tokyo

Allender, Friedman, and Gasarch proved an upper bound of PSPACE for the class DTTR of decidable languages that are polynomial-time truth-table reducible to the set of prefix-free Kolmogorov-random strings regardless of the universal machine used in the definition of Kolmogorov complexity. It is conjectured that DTTR in fact lies closer to BPP, a lower bound established earlier by Buhrman, Fortnow, Koucky, and Loff. It is also conjectured that we have similar bounds for the analogous class defined by plain Kolmogorov randomness. We discuss how these conjectures become more plausible when we impose restrictions on the lengths of the queries asked in the reduction. (Based on joint work with Shuichi Hirahara presented at MFCS 2014.)

Bennett deep sequences in recursion theory

Frank Stephan, National University of Singapore

Logical depth measures the depth of a set, identified with its characteristic function, in terms of how much normal Kolmogorov complexity improves with respect to any resource-bounded version of it. These notions can be based on the prefix-free and the plain Kolmogorov complexity and the results are in both cases different. While for prefix-free Kolmogorov complexity, there is a strong connection between logical depth and truth-table reducibility, such a connection fails for plain Kolmogorov complexity even with respect to many-one reducibility. The talk highlights various recursion-theoretic properties of sets which have a close connection to logical depth.

Introduction to generic case complexity

R. Downey, Victoria University of Wellington

This lecture gives an introduction to the notion of coarse and generic computability. These notions come from group theory but have developed a life of their own. A generic computation is a partial function which always gives the correct answer, and halts on a set of Borel density 1. Coarse computation can make errors but is allowed to only be wrong on a set of Borel density 0. The complexity theory is largely undeveloped.

Coarse and generic computability of the halting problem

L. Bienvenu, CNRS and University Paris 7

Most of the current research on coarse and generic computability looks at approximability of sets by computable sets w.r.t. lower density. I will discuss a few results on approximability of the halting problem (properly formulated) when upper density is used instead. (joint work with Damien Desfontaines and Alexander Shen).

Lebesgue density and effectively closed sets

Mushfeq Khan, University of Hawaii

Analyzing the effective content of the Lebesgue density theorem played a crucial role in some recent developments in algorithmic randomness, namely, the solutions of the ML-covering and ML-cupping problems. Two new classes of reals emerged from this analysis: the positive density points with respect to effectively closed (or Π^0_1) sets of reals, and a proper subclass, the density-one points. Bienvenu, Hölzl, Miller, and Nies have shown that the Martin-Löf random positive density points are exactly the ones that do not compute the halting problem. Treating this theorem as our starting point, we present several new results that shed light on how density, randomness, and computational strength interact.

Notes on the reverse mathematics for WWKL

Keita Yokoyama, JAIST

It is well-known that Reverse Mathematics for analysis has a strong connection with computable analysis. In fact, many theorems in Reverse Mathematics can be understood as reformulations of theorems in computable analysis. For example, the Cauchy-Peano theorem requires WKL because there exists a computable differential equation whose solution always computes a PA-degree. On the other hand, some theorems might look rather strange. In this talk, I will consider the following example from [1]: Over RCA_0, WWKL is equivalent to the assertion that every continuous function from $[0, 1]$ to $[0, 1]$ is Riemann integrable.

[1] S. Sanders and Y, The Dirac delta function in two settings of Reverse Mathematics, Arch. Math. Logic 51, pp: 99121, 2012.

Invariant measures concentrated on Erdos-Renyi graphs

Cameron Freer, MIT

New Collaborations and joint work

The below is a small selection of new collaborations and papers in preparation arising from work at the workshop. Many of the new collaborations have been fruitful and are ongoing. The following are in no particular order.

- Bienvenu, Greenberg, and Monin worked up a project involving *higher randomness*, where the notion of effective test becomes coarser and hence the randomness is stronger.
- A new initiative was begun between Greenberg, Monin and Kihara about higher genericity and lowness. Nothing is really known here.
- Greenberg, Miller and Nies worked out a sketch of a new result concerning the sets below both halves of a random. The halves must be relatively random by van Lambalgen's Theorem, and hence the sets must be K-trivial. The question is how much smaller than K-triviality the class is. Results so far show that this relates a cost function related to Chaitin's halting probability.
- Downey, Hölzl, Franklin and Ng began a new project about the Borel density of degrees.

  $$\gamma(A) = \sup\{r \mid A \text{ is coarsely computable at density } r\}.$$

  $$\Gamma(A) = \inf\{\gamma(X) \mid X \leq_T A\}.$$

  These notions are due to Carl Jockusch and researchers from Madison. That group proved that in the hyperimmune degrees the only possible value of Γ is 0. In the hyperimmune-free degrees, this value can be $\frac{1}{2}$ if A is either 1-random or computably traceable. This group seems to have shown that $\frac{1}{2}$ is possible in other cases, $\frac{1}{2}$ is possible without there being a real with $\gamma(X) = \frac{1}{2}$ in the degree, and that no uniform proof of another value seems possible. The work is definitely and ongoing new collaboration.
- Downey and Greenberg as part of an ongoing project developed ideas to possibly show that their new hierarchy based on approximations can contain totally ω-c.a. degrees with no maximal ones above them, whereas each level has a maximal totally ω^2 one.
- Freer and Reimann were able to work on a project related to made progress on a project that began earlier, on the topologies of graph that yield the Henson graph upon sampling. The Henson graph is a version of the Erdos-Renyi graph that avoids triangles.
- Allender and Watanabe began a new project on MINIMUM CIRCUIT SIZE PROBLEM(a topic spoken about by Allender). It is too soon to know if anything will come out of it. Perhaps this might relate to the Watanabe group's grand challenge in complexity.
- Downey and Miyabe began a new project looking at integer valued randomness. They ask about separations of the partial notion, and also the relationship between integer valued randomness and the integer notion

in the hyperimmune free degrees. Downey and Miyabe followed this up with further work at Miyabe's University in Tokyo following the meeting. Whilst Downey and Miyabe were part of the organizing team, they have not previously worked on a project together. It is not yet clear what the outcome will be.

- Day and Nandakumar began work relating to Orstein's theorem about Bernoulli shifts and Shannon Entropy. On the way to the meeting, Day gave a new proof of the classical Kolmogorov-Sinai Theorem using Kolmogorov complexity. This simplifies the known proof using algorithmic methods. The Nandakumar and the Wellington group attempting to arrange for Nandakumar to visit Wellington early in the new year. Funding is pending. The interactions of Kolmogorov complexity and Ergodic Theory seems very fruitful. Several talks at the meeting related to this, and progress is ongoing.
- Bienvenu, Day, and Reimann met on several occasions to discuss questions related to effective dimension and ergodic theory. It was discussed whether Day's above mentioned result, or type of techniques, could be extended to dynamical systems where the group of actions is amenable, using combinatorial characterisations of such groups. Another topic discussed was whether or not one can find a natural Kolmogorov-complexity-like characterisation of Renyi's entropy.
- Khan, Day and Miller began a project about Jan Reimann's question: can Kjos-Hanssen's results related to Hippocratic randomness be extended to ergodic measures? This project is currently being pursued. Khan and Miller finished some material on bushy trees; a paper is to be submitted.
- Nandakumar, Watanabe, and Reimann began a project about using Burrows-Wheeler Transform as the basis for a similarity metric. One of the issues we discussed is the aspect of symmetry of distance in this setting.
- Nandakumar and Miyabe began a project looking at the possibility of using integrable tests in resource-bounded settings. This could help in proving resource-bounded ergodicity.
- Nandakumar and Day discussed the possibility of effective versions of the Fürstenberg Multiple Recurrence Theorem.
- Barmpalias and Lewis-Pye continued their work on Schelling Segregation (recent FOCS) which uses martingale methods to analyse clustering behaviour. The finished a paper for the Proceedings of the Royal Society A. They also began a new project based around the Papadimitriou papers on why evolution has favored sexual reproduction.

There were few talks with carefully chosen topics, in a flexible time frame, which encouraged much discussion already during the talks. The following quote from one of the attendees may summarize the feeling of several workshop participants.

"The meeting was very nice, and relaxed. There was greater time devoted to discussions rather than talks - a format that I personally consider more productive. Kindly consider organizing more events with the same format."

Participants

- Eric Allender, Rutgers University
- George Barmpalias, Victoria University of Wellington
- Laurent Bienvenu, CNRS & Université Paris 7
- Adam Day, Victoria University of Wellington
- Cameron Freer, Massachusetts Institute of Technology
- Noam Greenberg, Victoria University of Wellington
- Rupert Hölzl, National University of Singapore
- Akitoshi Kawamura, University of Tokyo
- Mushfeq Khan, University of Hawaii
- Takayuki Kihara, Japan Advanced Institute of Science and Technology
- Selwyn Ng, Nanyang Technological University
- Jan Reimann, Pennsylvania State University
- Frank Stephan, National University of Singapore
- Kohtaro Tadaki, Chuo University
- Keita Yokoyama, Japan Advanced Institute of Science and Technology
- Joseph Miller, University of Wisconsin Madison
- Johanna Franklin, Hofstra University
- Andy Lewis-Pye, LSE
- Benoit Monin, Paris Diderot
- Satyadev Nandakumar, Indian Institute of Technology Kanpur