Verifiable and Reusable Metaprograms for Bayesian Inference

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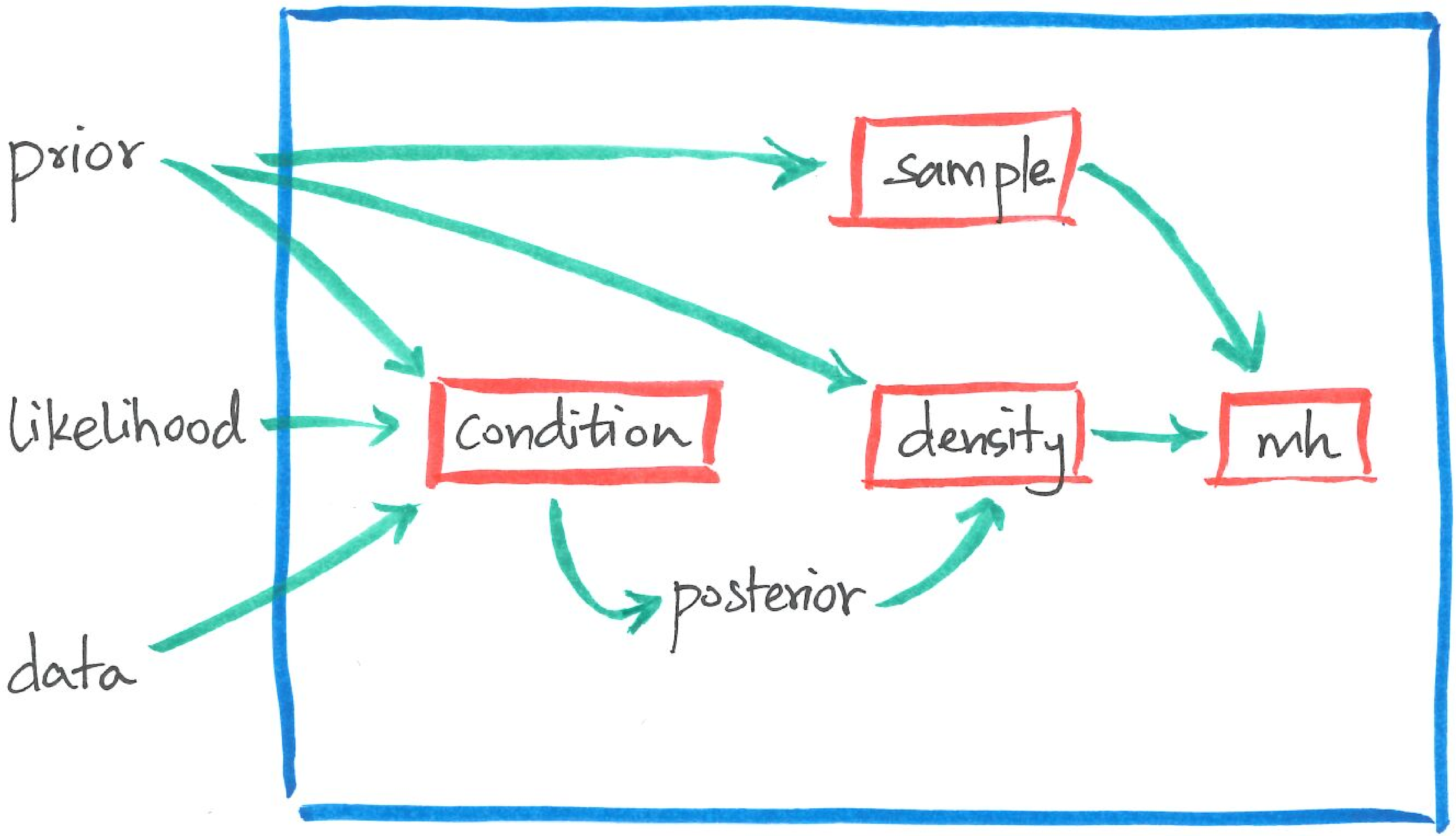
NII Shonan Meetings, May 2018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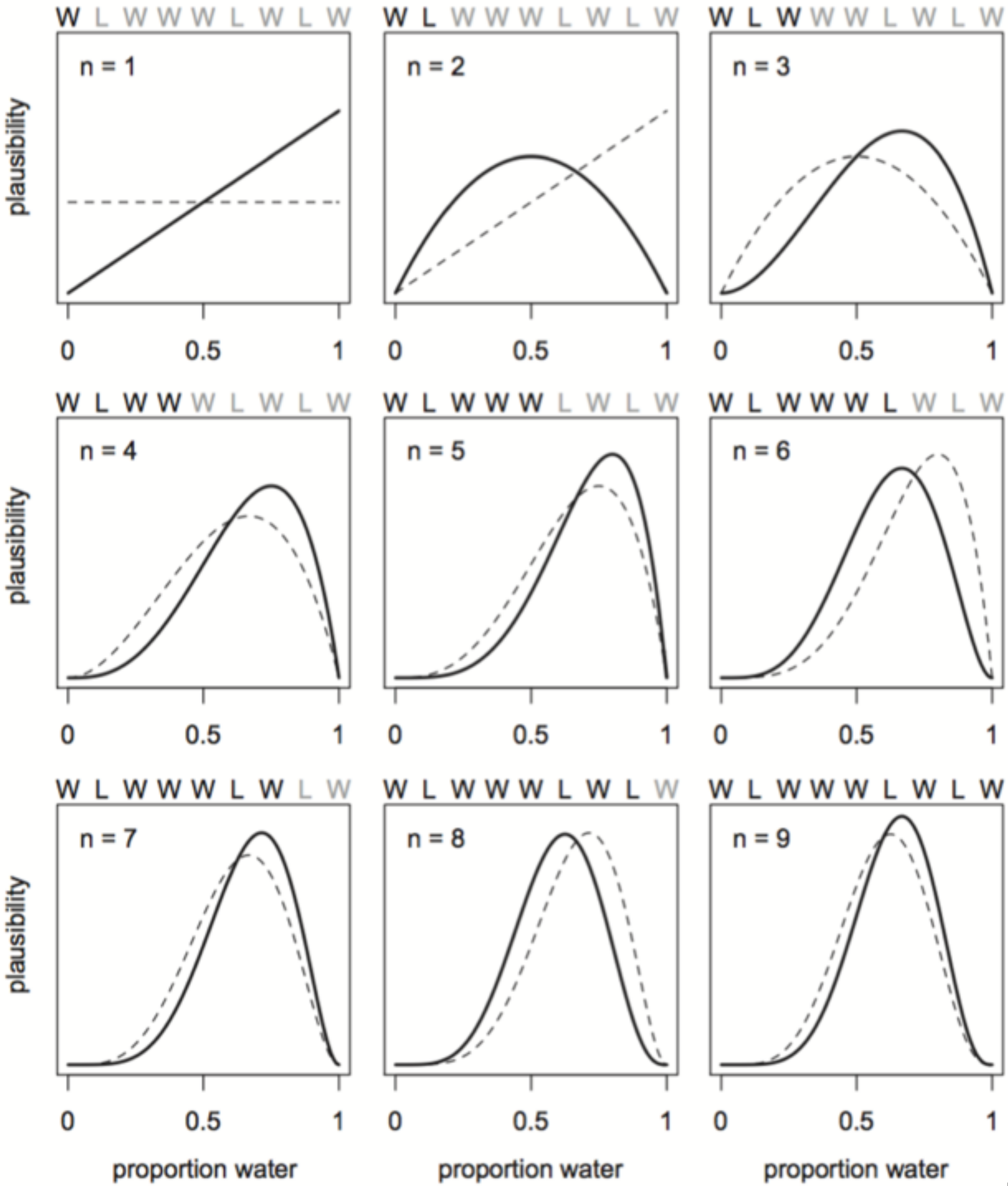

Richard McElreath, 2015

```
type R = Double
type Dist a = (a → R) → R
```

parameter

```
prior :: Dist R
prior f = integrate f 0 1
   where integrate = ...
```

parameter data

```
likelihood :: R → Dist Bool
likelihood p f = sum [if b then p * f b else (1-p) * f b
                      | b ← [True, False]]
```

Conditioning

```
posterior :: Bool -> Dist R
posterior dat f = prior (\p ->
                    likelihood p (\d ->
                    if d == dat then f p else 0))
                / marginal
  where marginal = prior (\p ->
                    likelihood p (\d ->
                    if d == dat then 1 else 0))
```

could be intractable

```
density :: Dist a → a → R
density d a = d (λa' → if a'==a then 1 else 0)

sample :: Dist a → IO a
sample d = invert (cdf d) <$> random 0 1
  where invert = ...
        cdf :: Dist a → a → R
        cdf d a = d (\a' → if a'<a then 1 else 0)
```

MCMC

```
mh :: Dist a → Dist a → IO [a]
mh target proposal = sample proposal >>= walk
  where walk :: a → IO [a]
        walk old =
          do new ← sample proposal
             u ← random 0 1
             let r = density target new * density proposal old
                     ------------------------------------------
                     density target old * density proposal new
                 s = if (min 1 r) < u then old else new
             (s:) <$> walk s
```

```
posterior :: Bool → Dist R
posterior d f = prior (\p →
                likelihood p (\d → ...))
```

exact

```
posterior' :: Bool → IO R
posterior' d = (/10^3) . sum . take 10^3 <$>
                 mh (posterior d) prior
```

approximate

MCMC

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                     ─────────────────────────────────────────
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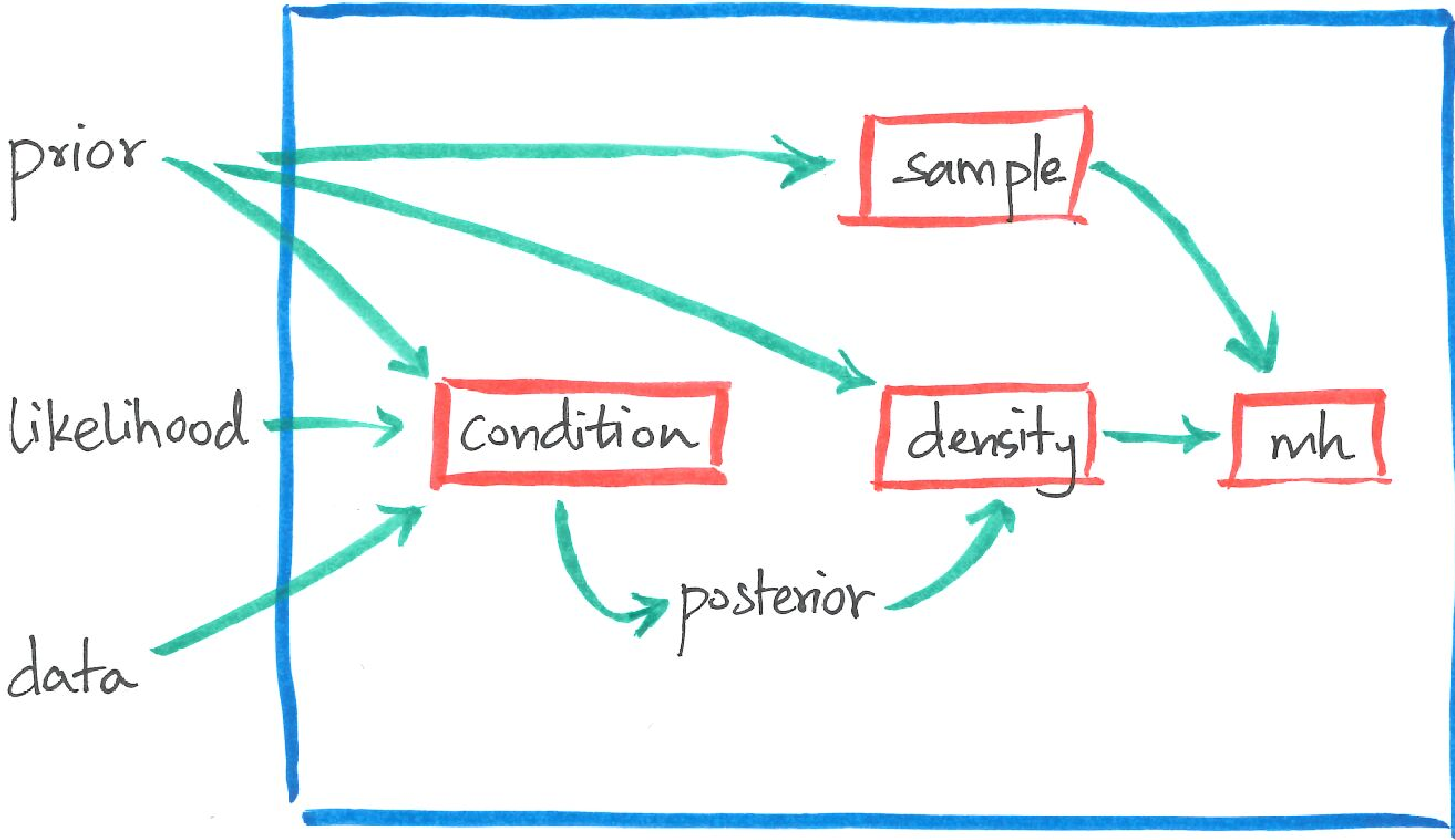
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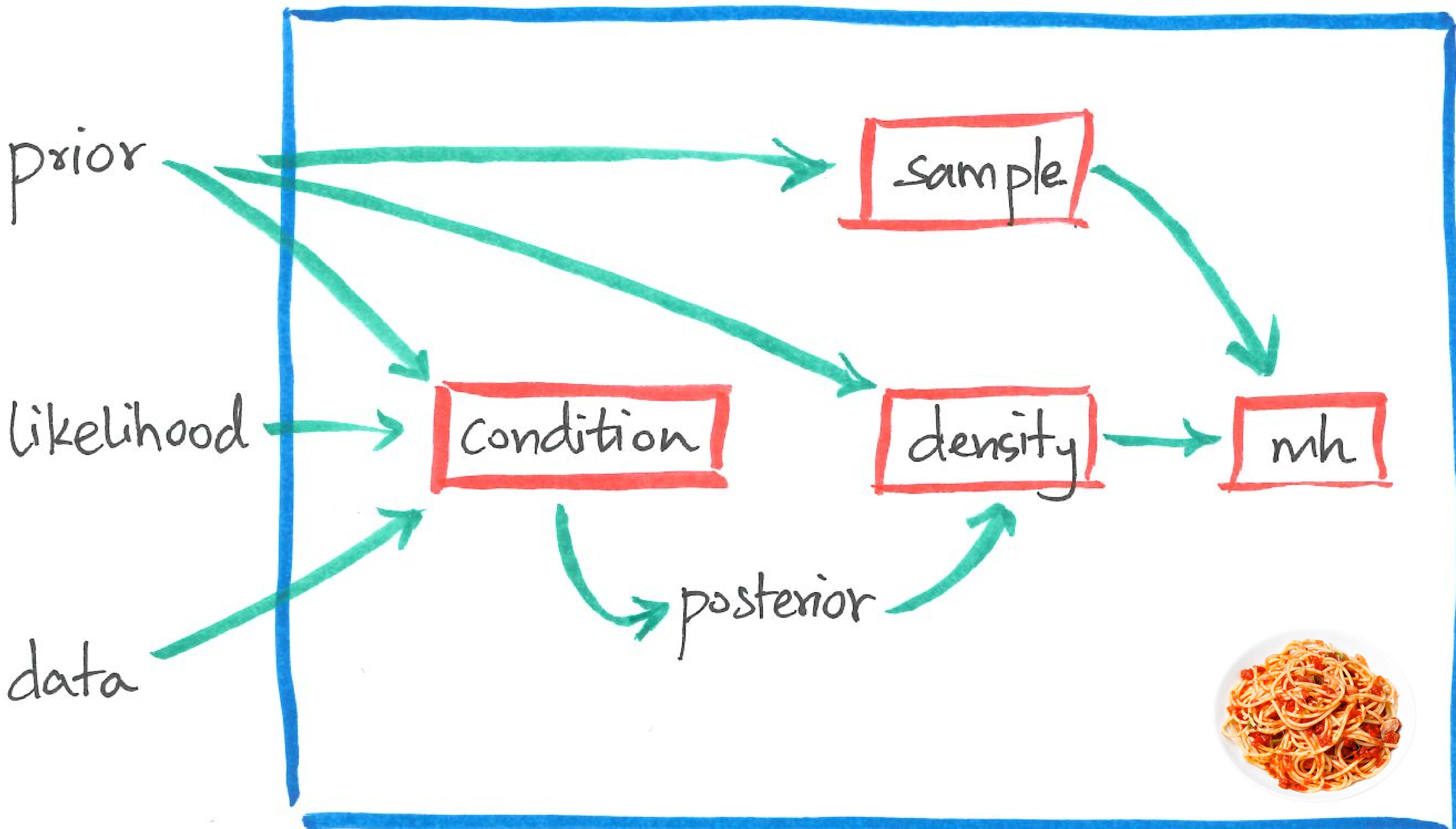
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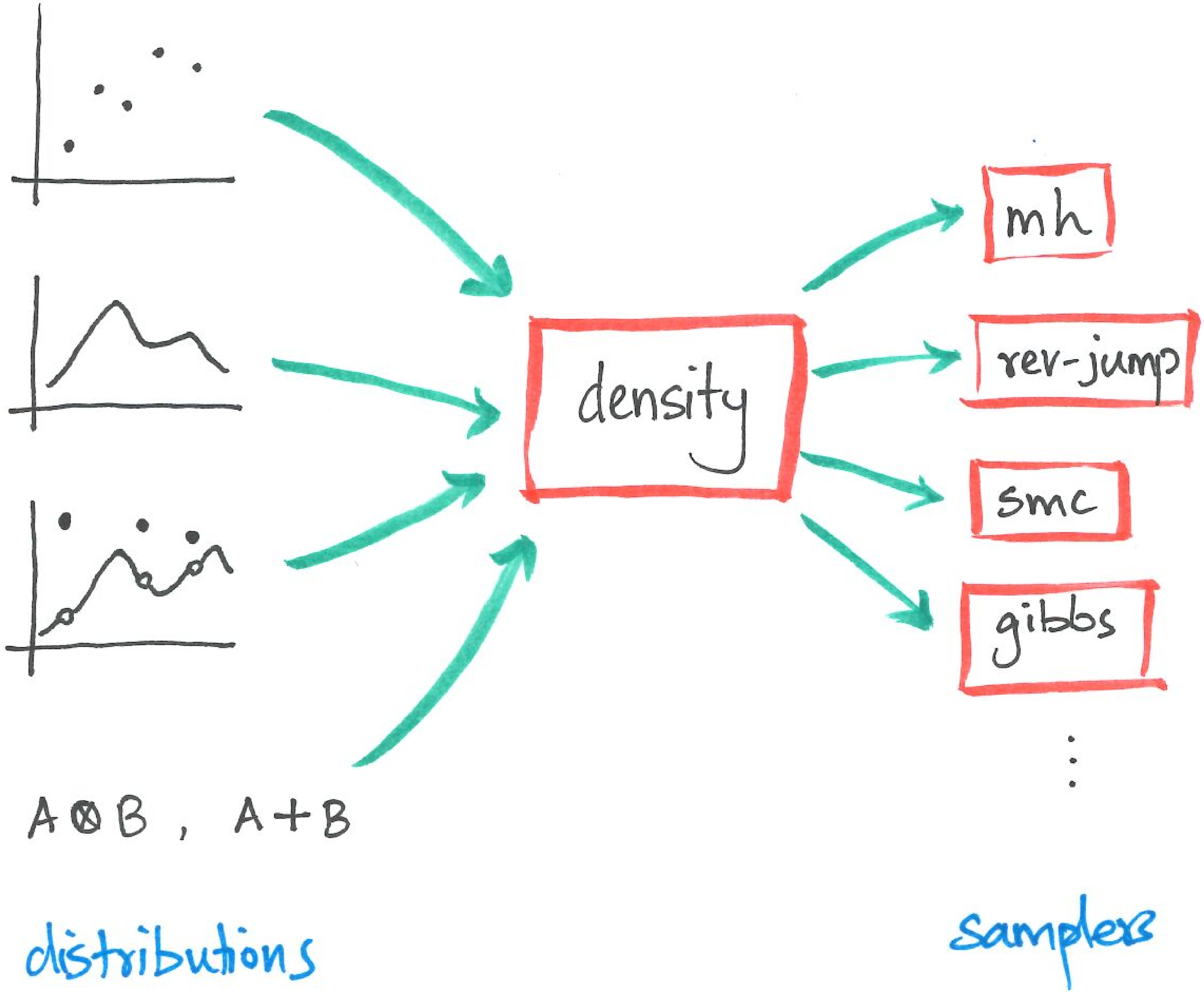

density
mh
rev-jump
smc
gibbs
A ⊗ B , A + B
distributions
samplers

$$[\![M\,a]\!] = ([\![a]\!] \to \mathbb{R}) \to \mathbb{R}$$

$$[\![\text{return } x]\!] = \lambda f.\ f\,x$$

$$[\![m \oplus m']\!] = \lambda f.\ [\![m]\!]f + [\![m']\!]f$$

$$[\![\text{do}\ \{\text{factor } w;\ m\}]\!] = \lambda f.\ w * [\![m]\!]f$$

example

likelihood :: $\mathbb{R} \to$ M Bool

likelihood p = do { factor p ; return true }

⊕ do { factor (1-p); return false }

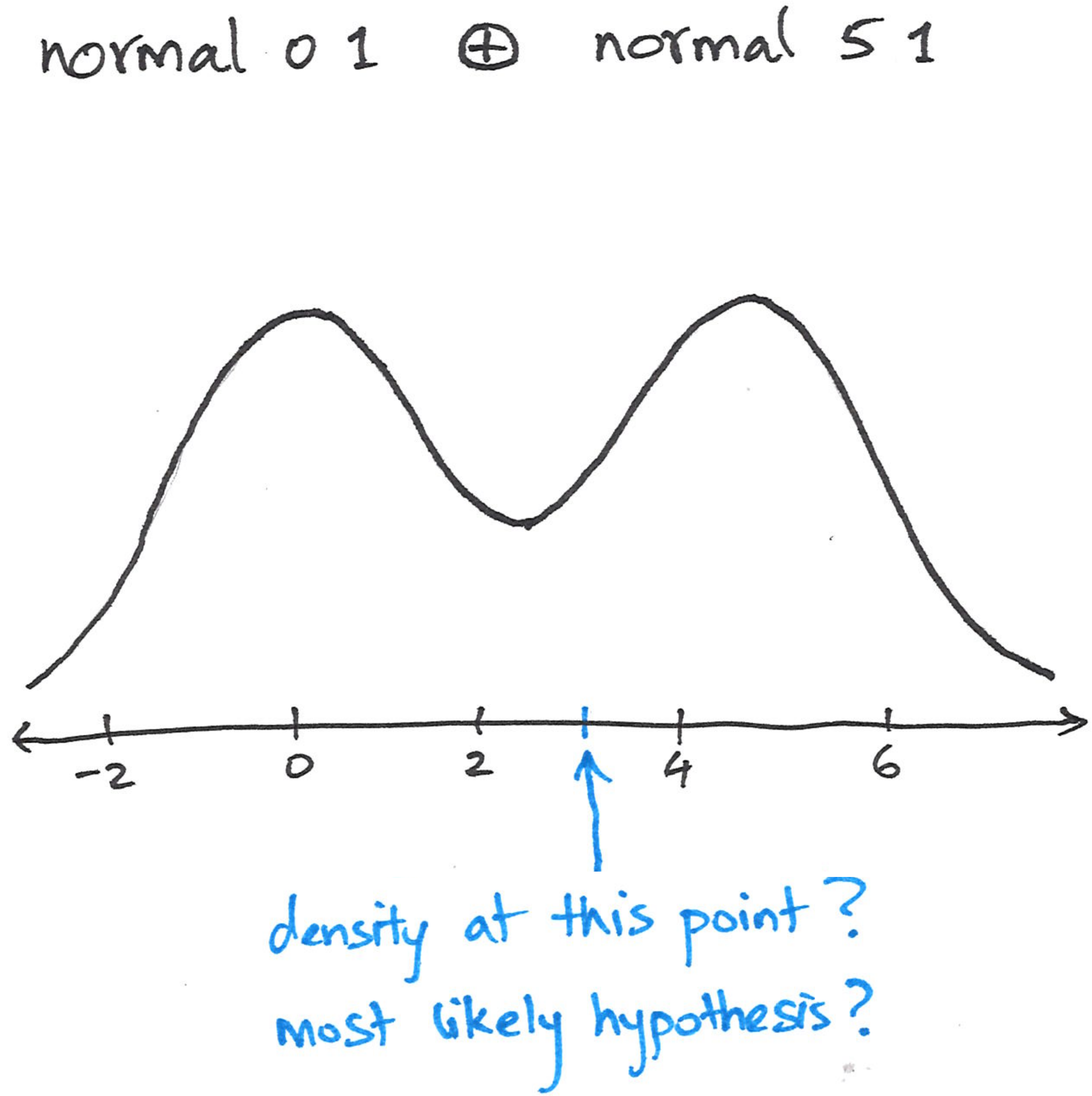
$$[\![\text{lebesgue}]\!] = \lambda f.\ \int_{-\infty}^{\infty} f\,x * dx$$

$$[\![\text{normal}\ \mu\ \sigma]\!] = \lambda f.\ \int_{-\infty}^{\infty} \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} * f\,x * dx$$

$$[\![\text{do}\ \{x \leftarrow m;\ m'\}]\!] = \lambda f.\ [\![m]\!]\ (\lambda t.\ [\![m']\!]\{x \mapsto t\}\ f)$$

example

```
prior :: M ℝ
prior = do {x ← lebesgue; observe (0<x<1); return x}
```



normal 0 1 ⊕ normal 5 1
-2
0
2
4
6
density at this point?
most likely hypothesis?

recall

$$\text{density} :: M\ a \to a \to \mathbb{R}$$

$$\text{density}\ m\ a = m\ (\lambda a' \to \text{if } a' == a \text{ then } 1 \text{ else } 0)$$

— executable specification

generalize

$$m\ f = b\ (\lambda a \to \text{density}\ m\ a * f\ a)$$

$$\text{Base}\ b ::= \text{counting} \mid \text{lebesgue}$$

$$[\![\text{counting}]\!] = \lambda f.\ \sum_{x \in \mathbb{N}} f\ x$$

$$[\![\text{lebesgue}]\!] = [\![\text{lebesgue}]\!] = \lambda f.\ \int_{-\infty}^{\infty} f\ x * dx$$

recall

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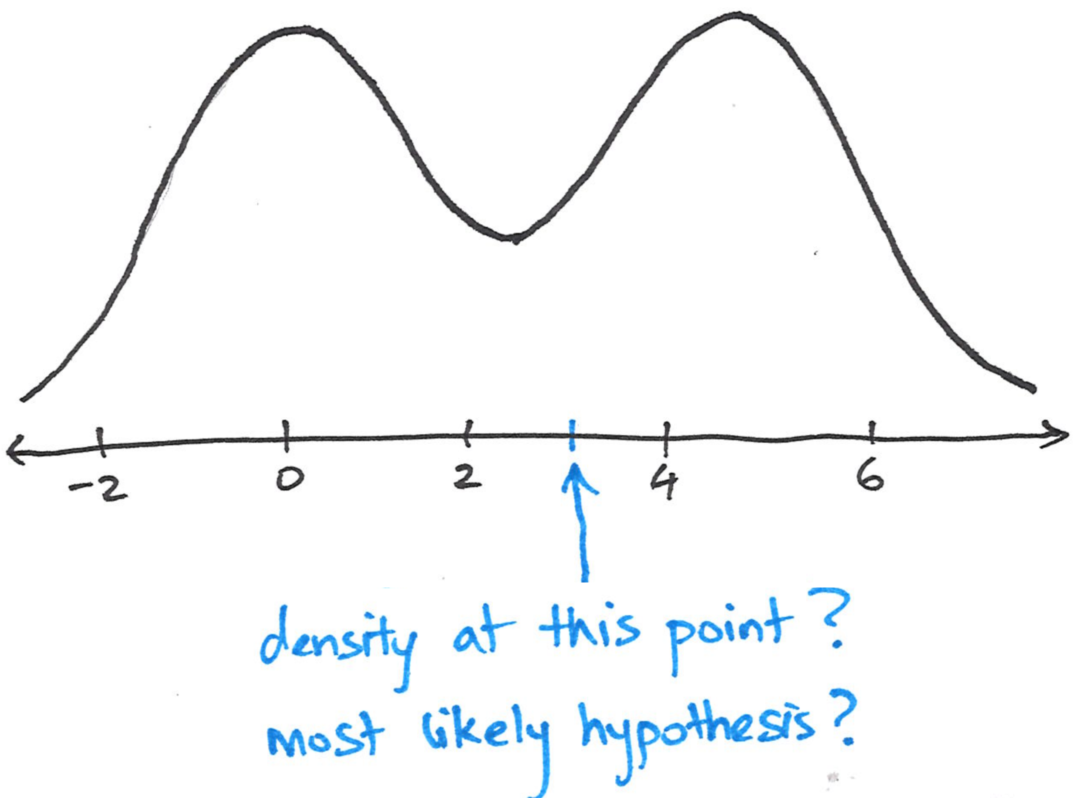
normal 0 1 ⊕ normal 5 1



d_1 = density (normal 0 1)

d_2 = density (normal 5 1)

$$m\ f = b\ (\lambda a \to \text{density}\ m\ a * f\ a)$$

Want d_1 = density (normal 0 1) such that:

$$[\![\text{normal}\ 0\ 1]\!]\ f = [\![\text{lebesgue}]\!]\ (\lambda a \to d_1\ a * f\ a)$$

$$m\ f = b\ (\lambda a \to \text{density}\ m\ a * f\ a)$$

Want d_1 = density (normal 0 1) such that:

$$[\![\text{normal } 0\ 1]\!]\ f = [\![\text{lebesgue}]\!]\ (\lambda a \to d_1\ a * f\ a)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} * f\ x * dx = \int_{-\infty}^{\infty} d_1\ x * f\ x * dx$$

$$\Rightarrow d_1\ x = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

$$m\ f = b\ (\lambda a \to \text{density}\ m\ a * f\ a)$$

Want d_1 = density (normal 0 1) such that:

$$[\![\text{normal } 0\ 1]\!]\ f = [\![\text{lebesgue}]\!]\ (\lambda a \to d_1\, a * f\, a)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} * f\, x * dx = \int_{-\infty}^{\infty} d_1\, x * f\, x * dx$$

$$\Rightarrow \quad d_1\, x = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

Similarly, d_2 = density (normal 5 1) = $\dfrac{e^{-(x-5)^2/2}}{\sqrt{2\pi}}$

$$m\ f = b\ (\lambda a \to \text{density}\ m\ a * f\ a)$$

And for the density of the mixture (Δ), we need:

$$[\![\text{normal}\ 0\ 1 \oplus \text{normal}\ 5\ 1]\!]\ f = [\![\text{lebesgue}]\!]\ (\lambda a \to \Delta\ a * f\ a)$$

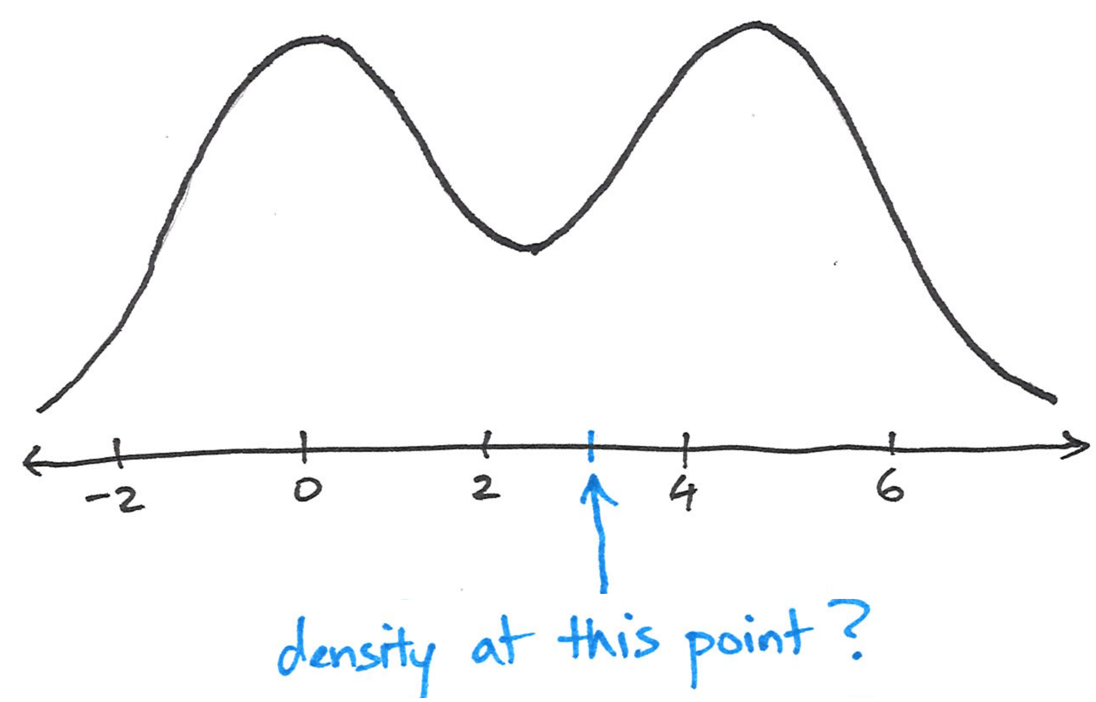
$$m\ f = b\ (\lambda a \to \text{density}\ m\ a * f\ a)$$

And for the density of the mixture (Δ), we need:

$$[\![\text{normal}\ 0\ 1 \oplus \text{normal}\ 5\ 1]\!]\ f = [\![\text{lebesgue}]\!]\ (\lambda a \to \Delta\ a * f\ a)$$

$$\Rightarrow \int_{-\infty}^{\infty} d_1 x * f\ x * dx + \int_{-\infty}^{\infty} d_2 x * f\ x * dx = \int_{-\infty}^{\infty} \Delta\ x * f\ x * dx$$

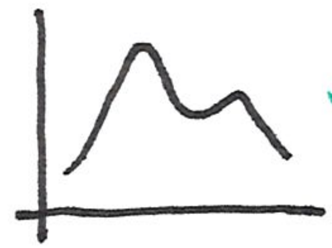
$$\Rightarrow \Delta\ x = d_1 x + d_2 x$$



$$m\ f = b\ (\lambda a \rightarrow \text{density}\ m\ a * f\ a)$$

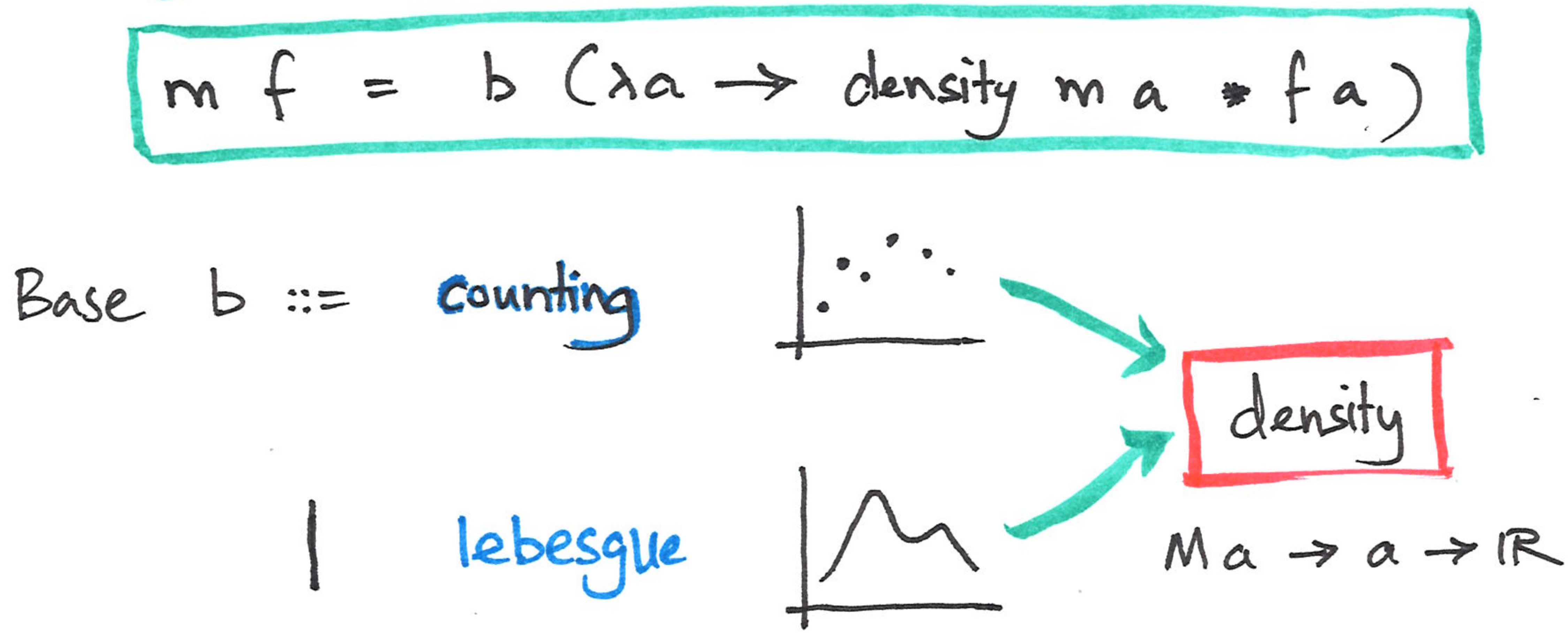
Base b ::= counting

| lebesgue



density

$M\ a \rightarrow a \rightarrow \mathbb{R}$



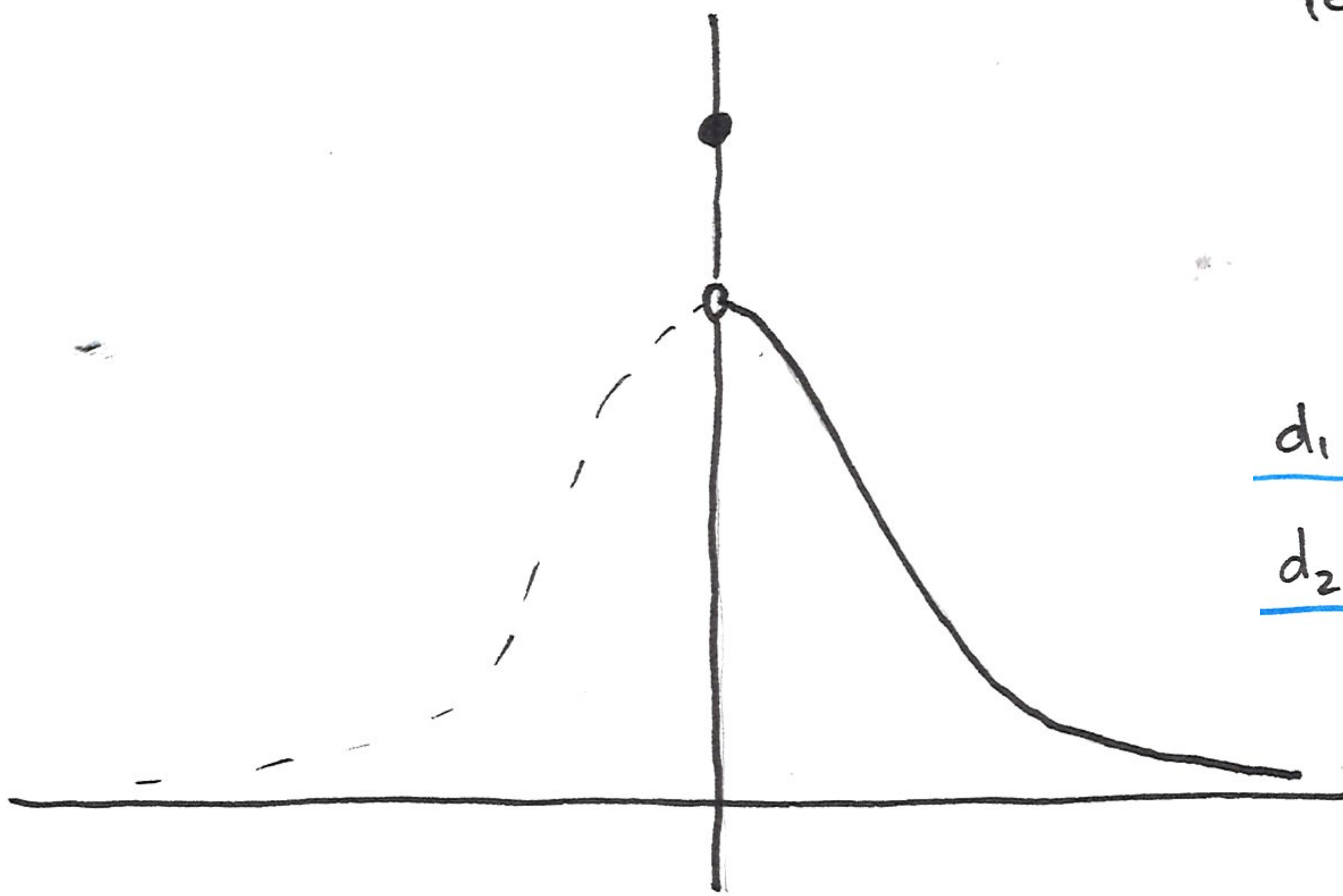
Deriving Probability Density Functions from Probabilistic Functional Programs

Sooraj Bhat[1], Johannes Borgström[2], Andrew D. Gordon[3,4], and Claudio Russo[3]

Deriving a Probability Density Calculator (Functional Pearl)

Wazim Mohammed Ismail   Chung-chieh Shan

```
clamped = do { x ← normal 0 1 ;        = do { x ← normal 0 1 ;          ] m1
               if x ≥ 0                       observe x ≥ 0 ;           ]
               then return x                  return x }                ]
               else return 0 }          ⊕ do { x ← normal 0 1 ;         ] m2
                                              observe x < 0 ;           ]
                                              return 0 }                ]
```



d_1 = density m_1

d_2 = density m_2

$$m\ f = b\ (\lambda a \to \text{density}\ m\ a * f\ a)$$

Want d_1 = density m_1 such that:

$$[\![\text{do}\ \{x \leftarrow \text{normal}\ 0\ 1;\ \text{observe}\ x \geq 0;\ \text{return}\ x\}]\!]\ f = [\![\text{lebesgue}]\!]\ (\lambda a \to d_1\ a * f\ a)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} * (\text{if } x \geq 0 \text{ then } 1 \text{ else } 0) * f\ x * dx = \int_{-\infty}^{\infty} d_1\ x * f\ x * dx$$

$$\Rightarrow d_1\ x = \text{if } x \geq 0 \text{ then } \frac{e^{-x^2/2}}{\sqrt{2\pi}} \text{ else } 0$$

Thus, m_1 <: lebesgue

"has a density wrt"

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Thus, m_1 <: lebesgue

"has a density wrt"

$$m\ f = b\ (\lambda a \to \text{density}\ m\ a * f\ a)$$

Want $d_2 = \text{density}\ m_2$ such that:

$$[\![\text{do}\ \{x \leftarrow \text{normal}\ 0\ 1;\ \text{observe}\ x<0;\ \text{return}\ 0\}]\!]\ f = [\![\text{lebesgue}]\!]\ (\lambda a \to d_2\, a * f\, a)$$

$$m\ f = b\ (\lambda a \to \text{density}\ m\ a * f\ a)$$

Want $d_2 = \text{density}\ m_2$ such that:

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$$\Rightarrow f\ 0 * \int_{-\infty}^{0} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\ dx = \int_{-\infty}^{\infty} d_2\, x * f x * dx$$

$$\Rightarrow f\ 0 * 1/2 = \int_{-\infty}^{\infty} d_2\, x * f\ x * dx$$

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this can never hold! consider $f\ x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$

So, $m_2 \not<:$ lebesgue

But, $m_2 <:$ return 0, where $[\![\text{return}\ x]\!] = [\![\text{return}\ x]\!] = \lambda f.\ f\ x$

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So, $m_2 \not<:$ lebesgue

But, m_2 <: return 0, where $[\![\text{return}\ x]\!] = [\![\text{return}\ x]\!] = \lambda f.\ f\ x$

To summarize,

clamped = $m_1 \oplus m_2$

m_1 <: lebesgue (<: means "has a density wrt")

m_2 ≮: lebesgue

m_2 <: return 0

clamped <: lebesgue ⊕ return 0

We need to extend our base measure language

Base b ::= counting | lebesgue | return e | b ⊕ b

$\mathbb{N}$

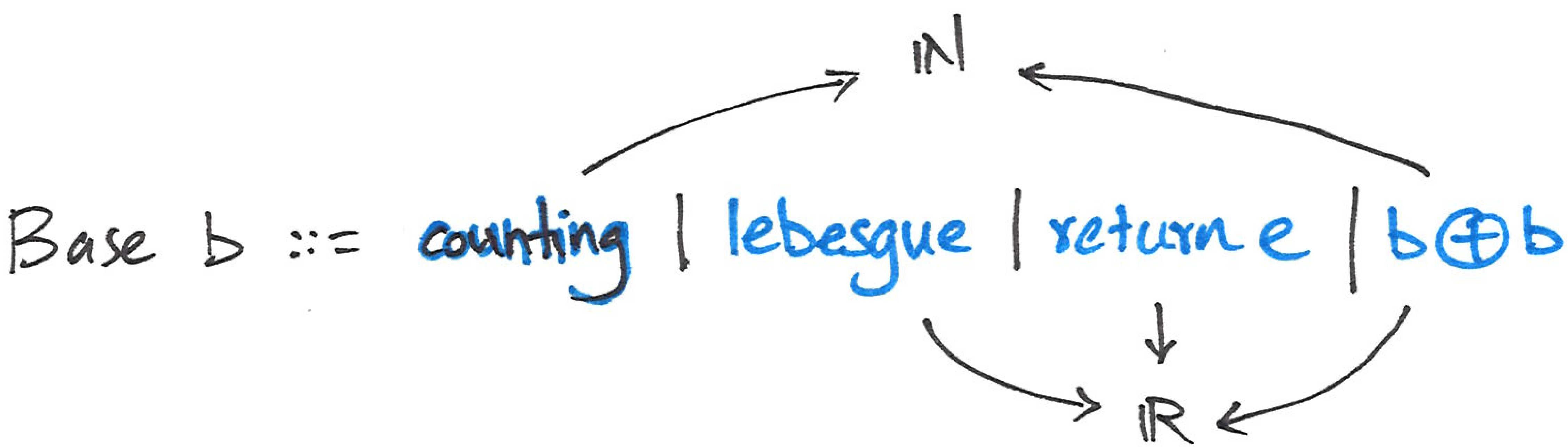
$$\text{Base } b ::= \text{counting} \mid \text{lebesgue} \mid \text{return } e \mid b \oplus b$$

$\mathbb{R}$

$$\text{density} :: \text{M } a \to \text{Base } a \to a \to \mathbb{R}$$

$$[\![m]\!]\, f = [\![b]\!]\, (\lambda a \to \text{density } m\, b\, a * f\, a)$$

$$m = \text{do } \{x \leftarrow b;\ \text{factor } (\text{density } m\, b\, x);\ \text{return } x\}$$



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$$\underline{\text{lebesgue}} = \text{do}\ \{ x \Leftarrow \text{lebesgue};\ \text{return}\ x \}$$

$$= \text{do}\ \{ x \Leftarrow \text{lebesgue};\ \text{observe}\ (x \neq 0);\ \text{return}\ x \}$$

$$= \text{do}\ \{ x \Leftarrow \text{lebesgue};\ \text{observe}\ (x \neq 0);\ \text{return}\ x \} \oplus \text{fail}$$

$$= \text{do}\ \{ x \Leftarrow \text{lebesgue};\ \text{observe}\ (x \neq 0);\ \text{return}\ x \} \oplus \text{do}\ \{ x \Leftarrow \text{return}\ 0;\ \text{observe}\ (x \neq 0);\ \text{return}\ x \}$$

$$= \text{do}\ \{ x \Leftarrow \underline{\text{lebesgue} \oplus \text{return}\ 0};\ \text{observe}\ (x \neq 0);\ \text{return}\ x \}$$

$$\text{density lebesgue}\ (\text{lebesgue} \oplus \text{return}\ 0)\ x = \text{if}\ x \neq 0\ \text{then}\ 1\ \text{else}\ 0$$

m = do { $x \leftarrow b$; factor (density m b x); return x }

m = lebesgue, b = lebesgue $\oplus$ return 0

lebesgue = do { $x \leftarrow$ lebesgue; return x }

= do { $x \leftarrow$ lebesgue; observe ($x \neq 0$); return x }

$m = \text{do}\ \{x \leftarrow b;\ \text{factor}\ (\text{density}\ m\ b\ x);\ \text{return}\ x\}$

$m = \text{lebesgue},\quad b = \text{lebesgue} \oplus \text{return}\ 0$

$$\begin{aligned}
\underline{\text{lebesgue}} &= \text{do}\ \{x \leftarrow \text{lebesgue};\ \text{return}\ x\} \\
&= \text{do}\ \{x \leftarrow \text{lebesgue};\ \text{observe}\ (x \neq 0);\ \text{return}\ x\} \\
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&\quad \oplus \text{fail} \\
&= \text{do}\ \{x \leftarrow \text{lebesgue};\ \text{observe}\ (x \neq 0);\ \text{return}\ x\} \\
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&= \text{do}\ \{x \leftarrow \underline{\text{lebesgue} \oplus \text{return}\ 0};\ \text{observe}\ (x \neq 0);\ \text{return}\ x\}
\end{aligned}$$

$\text{density lebesgue}\ (\text{lebesgue} \oplus \text{return}\ 0)\ x = \text{if}\ x \neq 0\ \text{then}\ 1\ \text{else}\ 0$

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&\quad \oplus \text{fail} \\
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&\quad \oplus \text{do}\ \{x \leftarrow \text{return}\ 0;\ \text{observe}\ (x \neq 0);\ \text{return}\ x\} \\
&= \text{do}\ \{x \leftarrow \text{lebesgue} \oplus \text{return}\ 0;\ \text{observe}\ (x \neq 0);\ \text{return}\ x\}
\end{aligned}$$

$$\text{density lebesgue}\ (\text{lebesgue} \oplus \text{return}\ 0)\ x = \text{if}\ x \neq 0\ \text{then}\ 1\ \text{else}\ 0$$

$$m = \text{do}\ \{x \leftarrow b;\ \text{factor}\ (\text{density}\ m\ b\ x);\ \text{return}\ x\}$$

$$m = \text{lebesgue},\quad b = \text{lebesgue} \oplus \text{return}\ 0$$

$$\underline{\text{lebesgue}} = \text{do}\ \{x \leftarrow \text{lebesgue};\ \text{return}\ x\}$$

$$= \text{do}\ \{x \leftarrow \text{lebesgue};\ \text{observe}\ (x \neq 0);\ \text{return}\ x\}$$

$$= \text{do}\ \{x \leftarrow \text{lebesgue};\ \text{observe}\ (x \neq 0);\ \text{return}\ x\} \oplus \text{fail}$$

$$= \text{do}\ \{x \leftarrow \text{lebesgue};\ \text{observe}\ (x \neq 0);\ \text{return}\ x\} \oplus \text{do}\ \{x \leftarrow \text{return}\ 0;\ \text{observe}\ (x \neq 0);\ \text{return}\ x\}$$

$$= \text{do}\ \{x \leftarrow \underline{\text{lebesgue} \oplus \text{return}\ 0};\ \text{observe}\ (x \neq 0);\ \text{return}\ x\}$$

$$\text{density lebesgue}\ (\text{lebesgue} \oplus \text{return}\ 0)\ x = \text{if}\ x \neq 0\ \text{then}\ 1\ \text{else}\ 0$$

$$m = \text{do}\ \{x \leftarrow b;\ \text{factor}\ (\text{density}\ m\ b\ x);\ \text{return}\ x\}$$

$$m = \text{lebesgue},\quad b = \text{lebesgue} \oplus \text{return}\ 0$$

$$\begin{aligned}
\text{lebesgue} &= \text{do}\ \{x \leftarrow \text{lebesgue};\ \text{return}\ x\} \\
&= \text{do}\ \{x \leftarrow \text{lebesgue};\ \text{observe}\ (x \neq 0);\ \text{return}\ x\} \\
&= \text{do}\ \{x \leftarrow \text{lebesgue};\ \text{observe}\ (x \neq 0);\ \text{return}\ x\} \\
&\quad \oplus \text{fail} \\
&= \text{do}\ \{x \leftarrow \text{lebesgue};\ \text{observe}\ (x \neq 0);\ \text{return}\ x\} \\
&\quad \oplus \text{do}\ \{x \leftarrow \text{return}\ 0;\ \text{observe}\ (x \neq 0);\ \text{return}\ x\} \\
&= \text{do}\ \{x \leftarrow \text{lebesgue} \oplus \text{return}\ 0;\ \text{observe}\ (x \neq 0);\ \text{return}\ x\}
\end{aligned}$$

$$\text{density lebesgue}\ (\text{lebesgue} \oplus \text{return}\ 0)\ x = \text{if}\ x \neq 0\ \text{then}\ 1\ \text{else}\ 0$$

$m = \text{do}\ \{ x \leftarrow b;\ \text{factor}\ (\text{density}\ m\ b\ x);\ \text{return}\ x \}$

$m = \text{return}\ 0, \quad b = \text{lebesgue} \oplus \text{return}\ 0$

$m = \text{do}\ \{x \leftarrow b;\ \text{factor}\ (\text{density}\ m\ b\ x);\ \text{return}\ x\}$

$m = \text{return}\ 0\ ,\quad b = \text{lebesgue} \oplus \text{return}\ 0$

$\underline{\text{return}\ 0} \ = \ \text{do}\ \{x \leftarrow \text{return}\ 0\ ;\ \text{observe}\ x = 0\ ;\ \text{return}\ x\}$

$m =$ do { $x \leftarrow b$; factor (density m b x); return x }

$m =$ return 0 , $b =$ lebesgue ⊕ return 0

return 0 = do { $x \leftarrow$ return 0 ; observe $x = 0$; return x }

= do { $x \leftarrow$ return 0 ; observe $x = 0$; return x }

⊕ fail

```
m = do {x ← b; factor (density m b x); return x}
```

```
m = return 0 ,   b = lebesgue ⊕ return 0
```

```
return 0  =  do {x ← return 0; observe x=0; return x}
          =  do {x ← return 0; observe x=0; return x}
             ⊕ fail
          =  do {x ← return 0; observe x=0; return x}
             ⊕ do {x ← lebesgue; observe x=0; return x}
```

$m = \text{do}\ \{x \leftarrow b;\ \text{factor}\ (\text{density}\ m\ b\ x);\ \text{return}\ x\}$

$m = \text{return}\ 0, \quad b = \text{lebesgue} \oplus \text{return}\ 0$

$$\begin{aligned}
\underline{\text{return}\ 0} &= \text{do}\ \{x \leftarrow \text{return}\ 0;\ \text{observe}\ x = 0;\ \text{return}\ x\} \\
&= \text{do}\ \{x \leftarrow \text{return}\ 0;\ \text{observe}\ x = 0;\ \text{return}\ x\} \\
&\quad \oplus \text{fail} \\
&= \text{do}\ \{x \leftarrow \text{return}\ 0;\ \text{observe}\ x = 0;\ \text{return}\ x\} \\
&\quad \oplus \text{do}\ \{x \leftarrow \text{lebesgue};\ \text{observe}\ x = 0;\ \text{return}\ x\} \\
&= \text{do}\ \{x \leftarrow \underline{\text{lebesgue} \oplus \text{return}\ 0};\ \text{observe}\ x = 0;\ \text{return}\ 0\}
\end{aligned}$$

$m = \text{do} \{x \leftarrow b;\ \text{factor}\ (\text{density}\ m\ b\ x);\ \text{return}\ x\}$

$m = \text{return}\ 0, \quad b = \text{lebesgue} \oplus \text{return}\ 0$

$$
\begin{aligned}
\text{return}\ 0 &= \text{do} \{x \leftarrow \text{return}\ 0;\ \text{observe}\ x=0;\ \text{return}\ x\} \\
&= \text{do} \{x \leftarrow \text{return}\ 0;\ \text{observe}\ x=0;\ \text{return}\ x\} \\
&\quad \oplus \text{fail} \\
&= \text{do} \{x \leftarrow \text{return}\ 0;\ \text{observe}\ x=0;\ \text{return}\ x\} \\
&\quad \oplus \text{do} \{x \leftarrow \text{lebesgue};\ \text{observe}\ x=0;\ \text{return}\ x\} \\
&= \text{do} \{x \leftarrow \text{lebesgue} \oplus \text{return}\ 0;\ \text{observe}\ x=0;\ \text{return}\ 0\}
\end{aligned}
$$

$\text{density}\ (\text{return}\ 0)\ (\text{lebesgue} \oplus \text{dirac}\ 0) = \text{if}\ x=0\ \text{then}\ 1\ \text{else}\ 0$

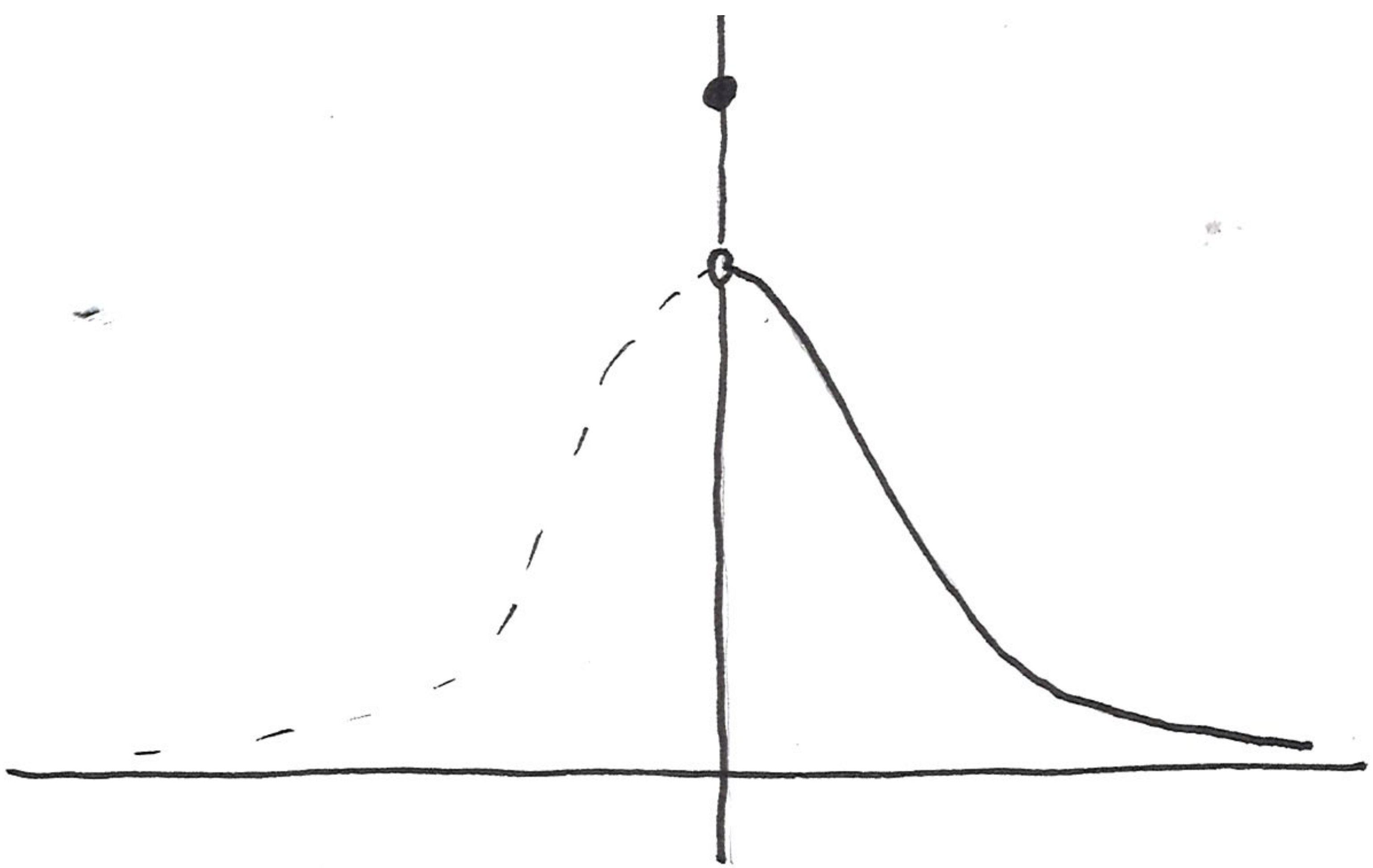
density :: M a → Base a → a → ℝ

m = do { x ⇜ b; factor (density m b x); return x }

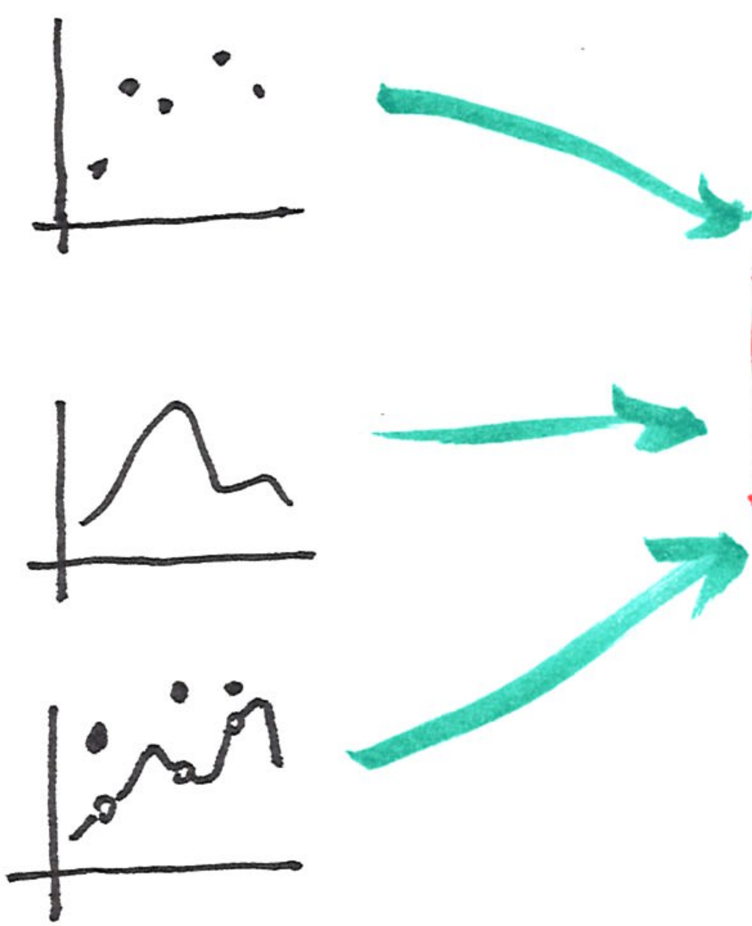
density lebesgue (lebesgue ⊕ return 0) x = if x ≠ 0 then 1 else 0

density (return 0) (lebesgue ⊕ dirac 0) = if x=0 then 1 else 0

Demo!



$$\text{Base } b ::= \text{counting} \mid \text{lebesgue} \mid \text{return } e \mid b \oplus b$$



$$\text{density} :: M\,a \to \text{Base } a \to a \to \mathbb{R}$$

Automatically inferring bases

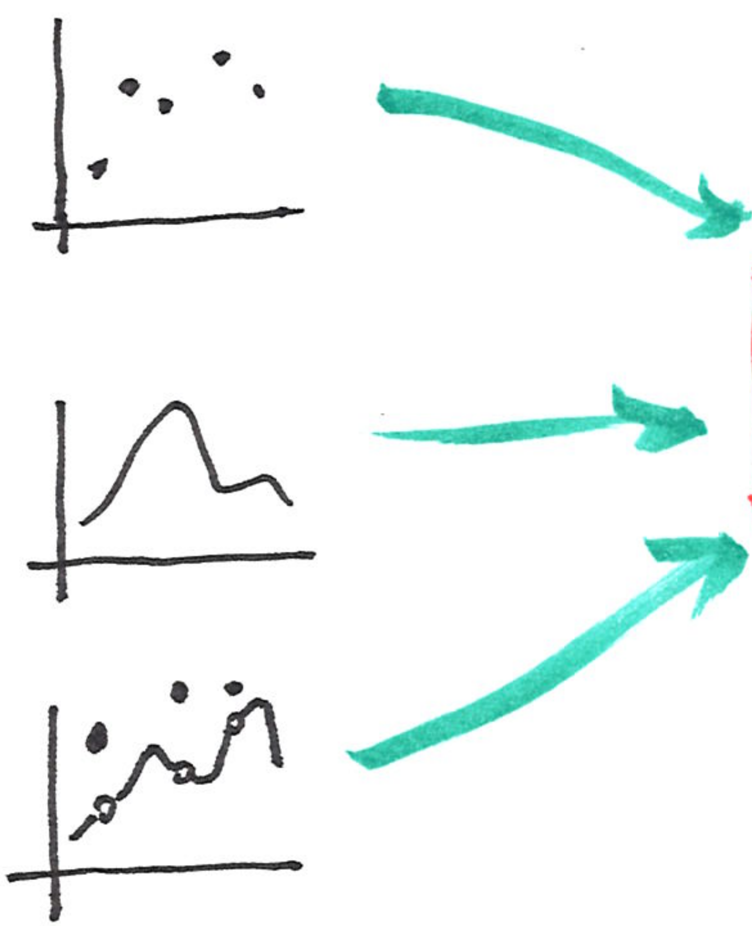
- When given a program, produce a set of density constraints

$$\left(\begin{array}{l}\textbf{do } \{x \leftsquigarrow \textbf{normal } 0\ 1; \\ \quad \textbf{observe } (0 \le x); \\ \quad \textbf{return } x\}\end{array}\right) \oplus \left(\begin{array}{l}\textbf{do } \{x \leftsquigarrow \textbf{normal } 0\ 1; \\ \quad \textbf{observe } (x < 0); \\ \quad \textbf{return } 0\}\end{array}\right) \longrightarrow \begin{array}{l}\textbf{lebesgue} <: b \\ \textbf{return } 0 <: b\end{array}$$

Demo!

$$\text{Base } b ::= \text{counting} \mid \text{lebesgue} \mid \text{return } e \mid b \oplus b$$

$$\mid\ b + b \ \mid\ \text{do } \{ x \leftarrow b;\ y \leftarrow b;\ \text{return } (x,y) \}$$



$$\text{density} :: M\,a \to \text{Base } a \to a \to \mathbb{R}$$

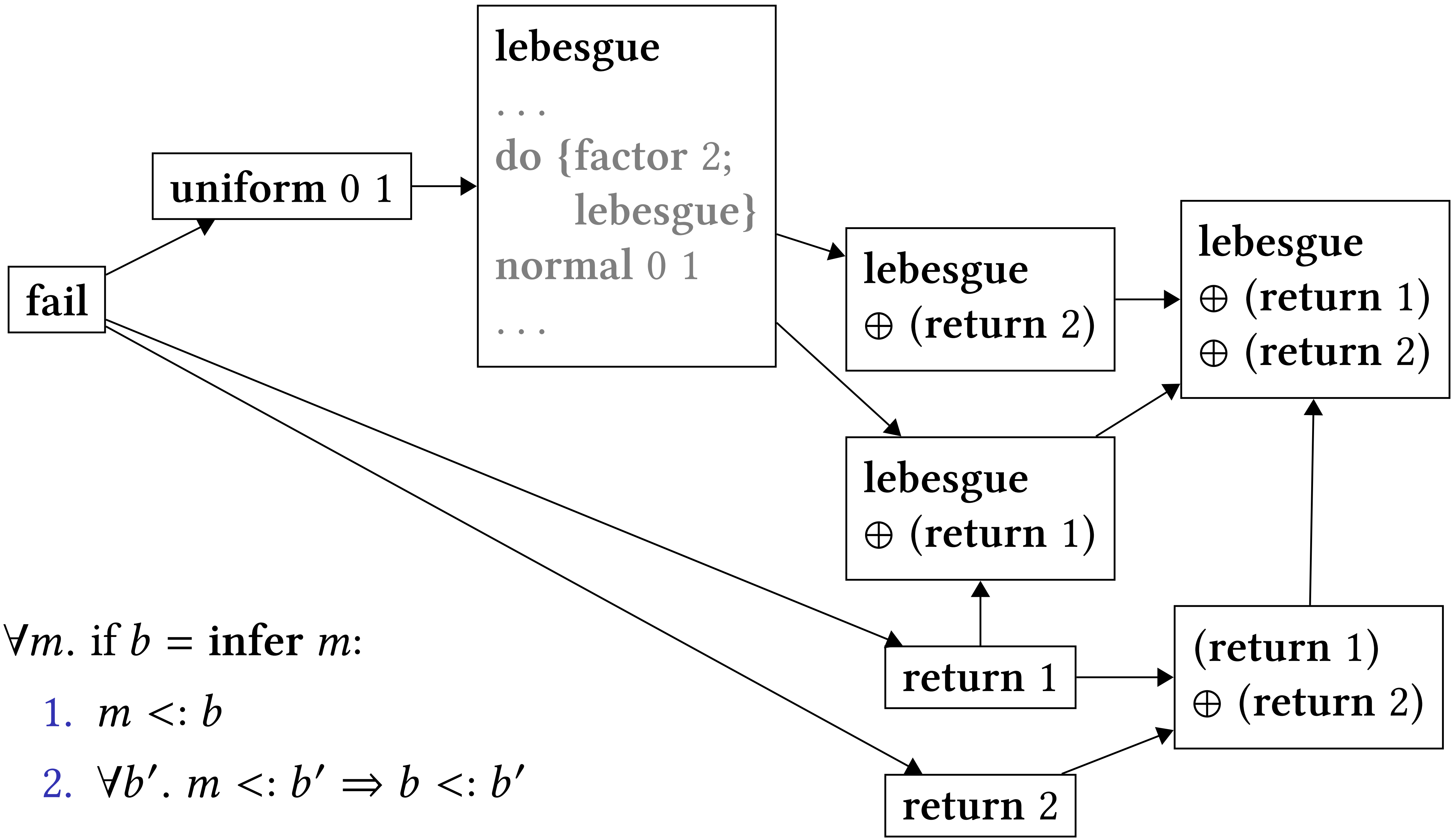
Automatically inferring bases

- When given a program, produce a set of density constraints

$$\begin{pmatrix}\textbf{do } \{x \leftharpoondown \textbf{normal } 0\ 1; \\ \quad \textbf{observe } (0 \le x); \\ \quad \textbf{return } x\}\end{pmatrix} \oplus \begin{pmatrix}\textbf{do } \{x \leftharpoondown \textbf{normal } 0\ 1; \\ \quad \textbf{observe } (x < 0); \\ \quad \textbf{return } 0\}\end{pmatrix} \longrightarrow \begin{array}{l}\textbf{lebesgue} <: b \\ \textbf{return } 0 <: b\end{array}$$

- Try to solve these constraints to produce a principal base measure
- $\forall m.$ if $b = \textbf{infer } m$:
  1. $m <: b$
  2. $\forall b'.\ m <: b' \Rightarrow b <: b'$

Density is a partial order

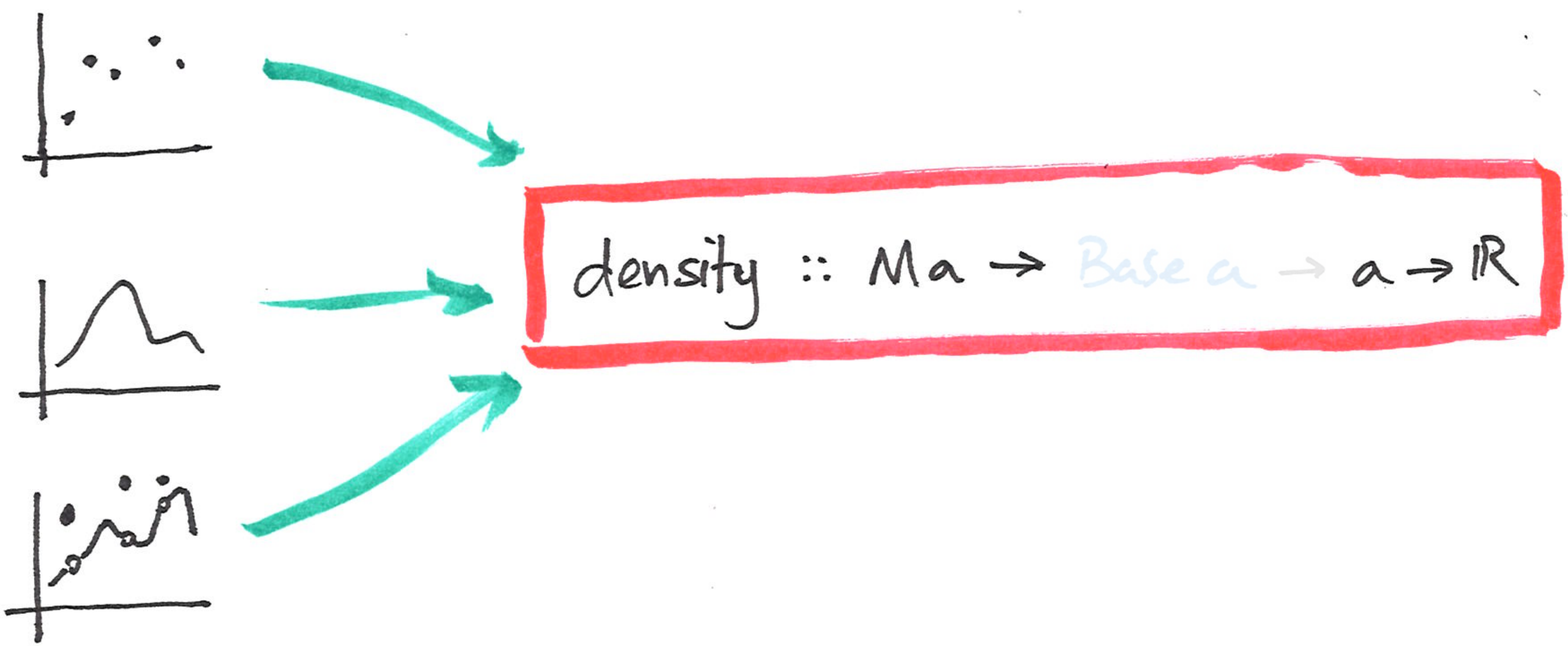


$\forall m.$ if $b =$ **infer** m:

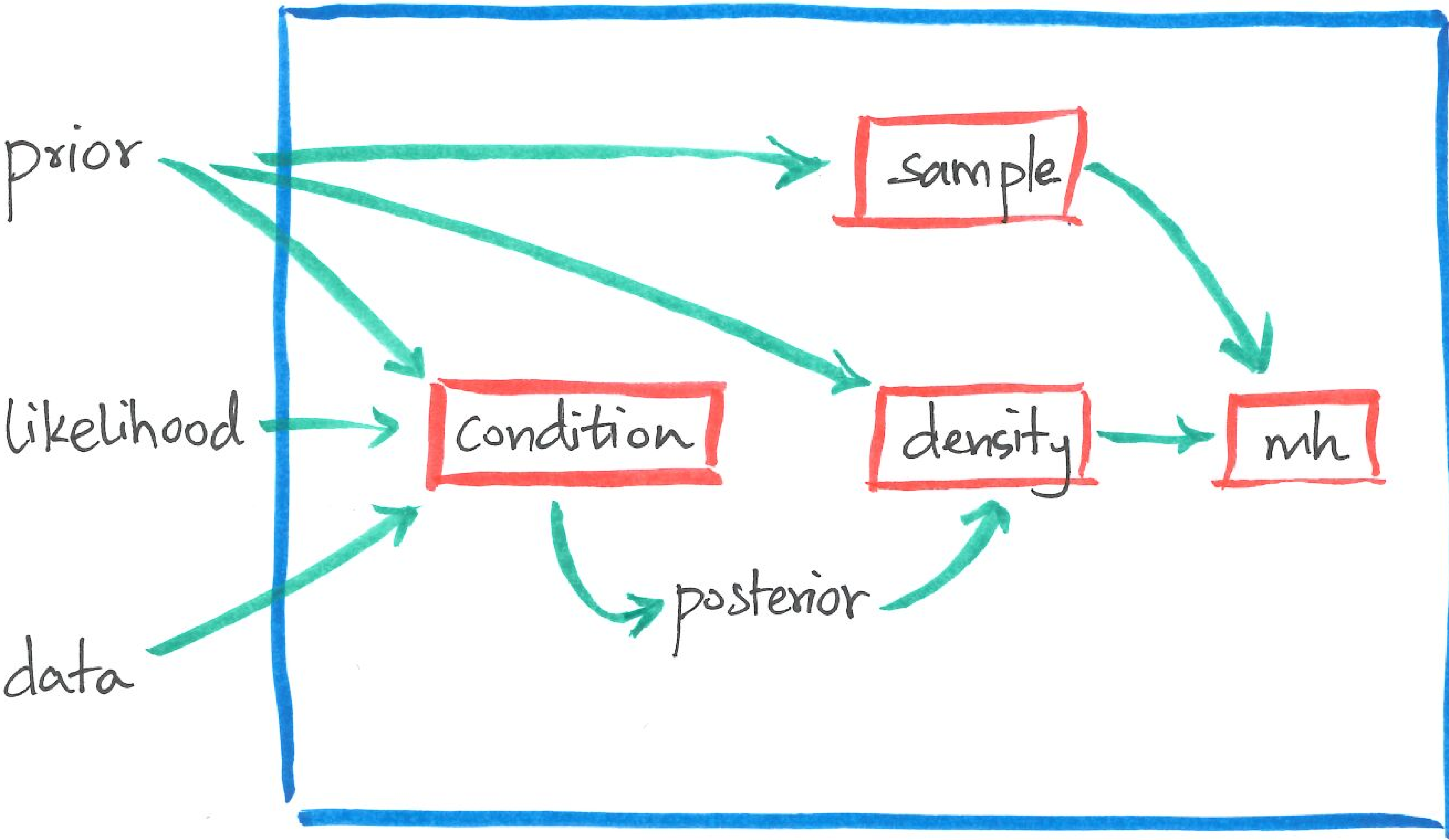
1. $m <: b$
2. $\forall b'.\ m <: b' \Rightarrow b <: b'$

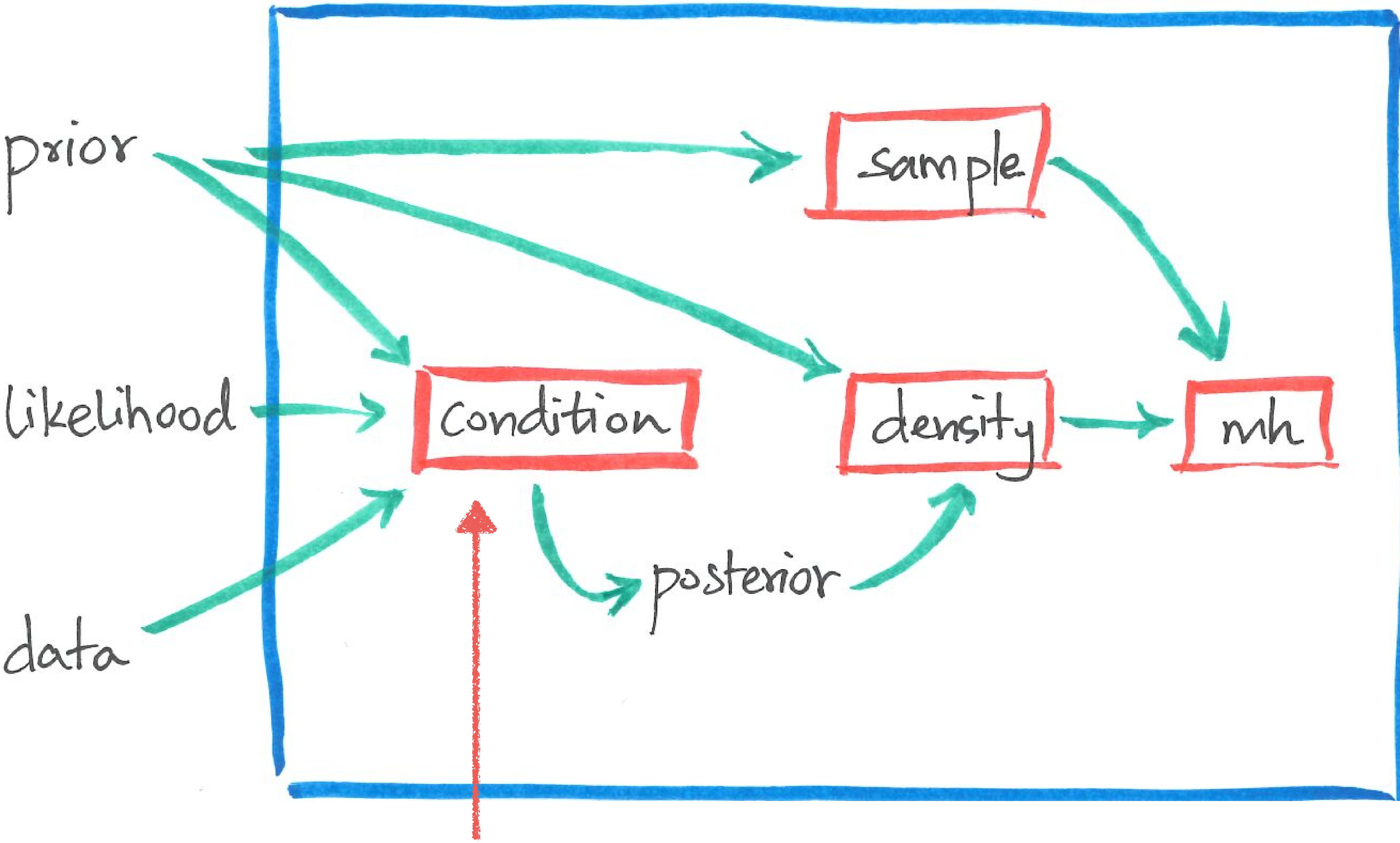
Base b ::= counting | lebesgue | return e | b ⊕ b

| b + b | do { x ← b; y ← b; return (x,y) }



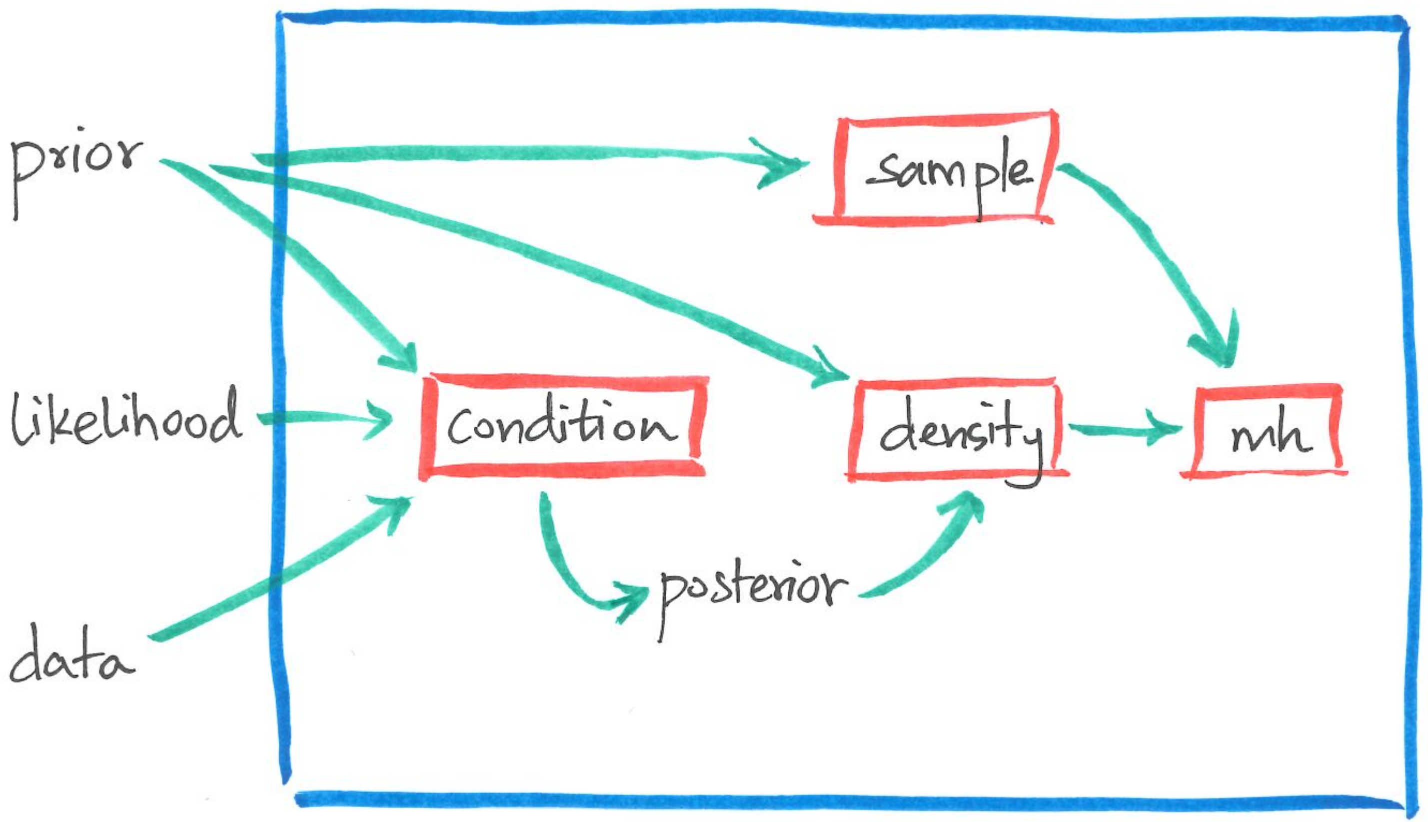
https://github.com/pravnar/disintegrating-mixtures


prior
likelihood
data
sample
condition
density
mh
posterior



Exact Bayesian Inference by Symbolic Disintegration

Chung-chieh Shan
Norman Ramsey



More support for symbolic disintegration

Praveen Narayanan   Chung-chieh Shan

PPS, POPL '18

Symbolic Conditioning of Arrays in Probabilistic Programs

PRAVEEN NARAYANAN, Indiana University, USA
CHUNG-CHIEH SHAN, Indiana University, USA

ICFP '17

Probabilistic inference by program transformation in Hakaru (system description)*

Praveen Narayanan[1], Jacques Carette[2], Wren Romano[1], Chung-chieh Shan[1], and Robert Zinkov[1]

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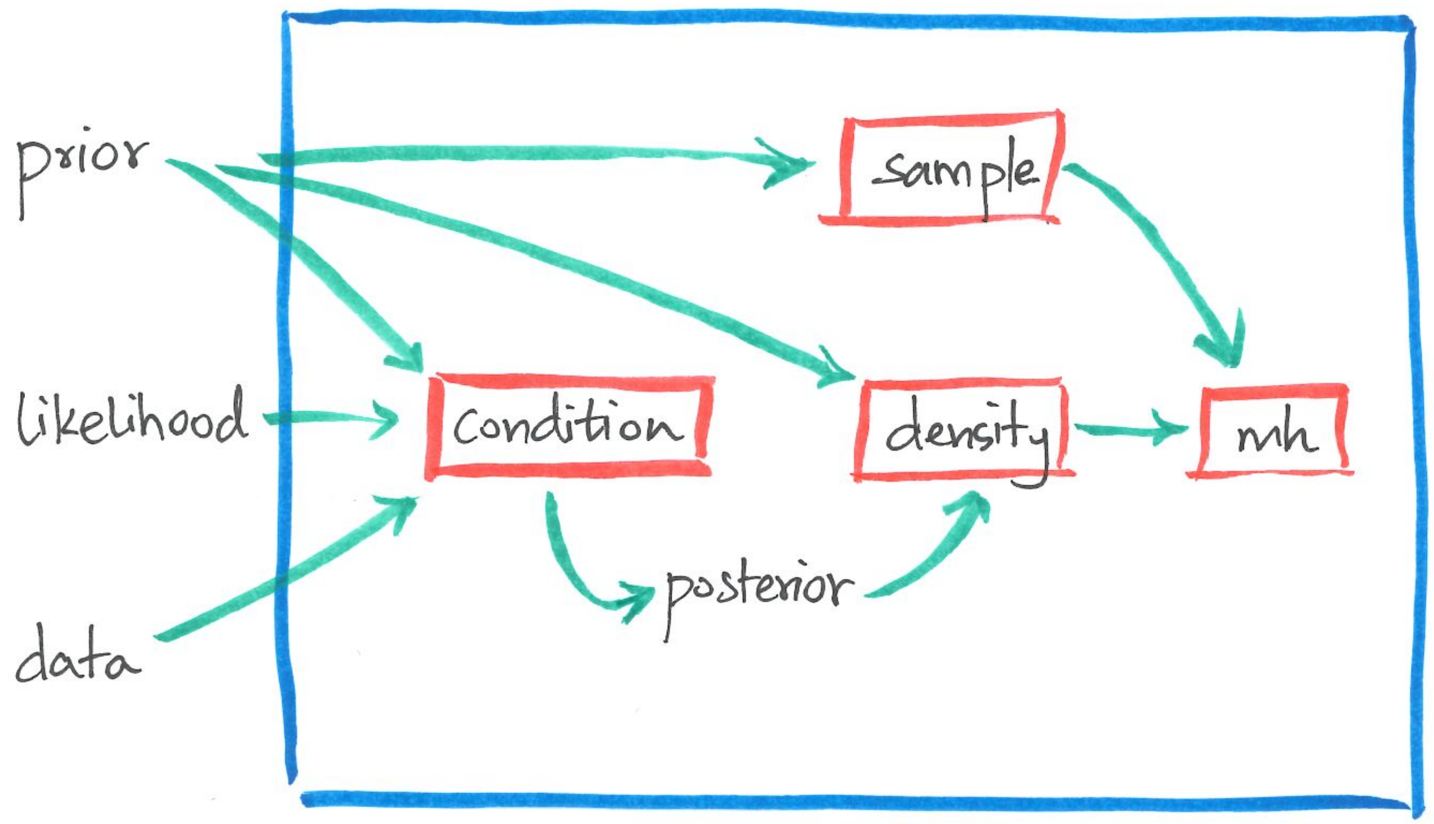

Thank you!

More support for symbolic disintegration

Praveen Narayanan   Chung-chieh Shan

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