Definability and Complexity of Counting Generalized Colorings:

New Results and Challenges

Johann A. Makowsky

Faculty of Computer Science,
Technion - Israel Institute of Technology,
Haifa, Israel
e-mail: janos@cs.technion.ac.il
http://www.cs.technion.ac.il/∼janos

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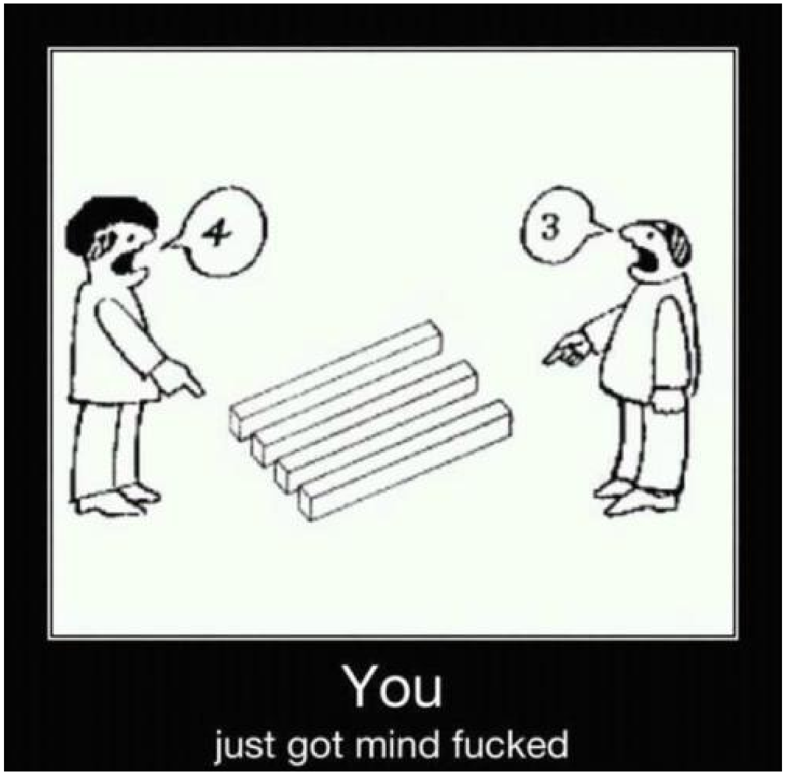
Graph polynomial project:
http://www.cs.technion.ac.il/∼janos/RESEARCH/gp-homepage.html

The Difficulty of Counting



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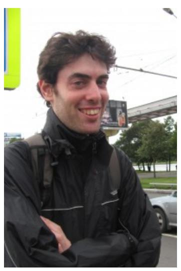
This is about work in progress

Joint work with

A. Goodall (Prague), M. Hermann (Paris), T. Kotek (Vienna),

S. Noble (London) and E.V. Ravve (Karmiel)

and my new graduate student V. Rakita.





T. Kotek M. Hermann S. Noble A. Goodall E. Ravve

Work done partially at the Simons Institute, Berkeley, 2016

Based partially on:

- A. Goodall, M. Hermann, T. Kotek, JAM, and S.D. Noble.
  On the complexity of generalized chromatic polynomials.
  Advances in Applied Mathematics, special issue on the Tutte polynomial, 2018
  Electronically available at `http://dx.do.org/10.1016/j.aam.2017.04.005`

- T. Kotek, JAM, and E.V. Ravve
  On sequences of polynomials arising from graph invariants
  European Journal of Combinatorics, 67 (2018), pp. 181-198

- JAM, E.V. Ravve and T. Kotek
  A logician's view of graph polynomials
  arXiv preprint arXiv:1703.02297

Outline

Part I: Harary's $\mathcal{P}$-colorings.

Part II (i): The complexity of the chromatic polynomial.

Part II (ii): Issues with the computational model.

Part II (iii): The Difficult Point Property (a former conjecture).

Part III: The complexity of univariate graph polynomials.

Conclusions: Challenges and summary.

Part I:

Harary's $\mathcal{P}$-colorings

and

generalized chromatic polynomials

Back to overview

The chromatic polynomial

Let G be a graph and $[k] = \{1, 2, \ldots, k\}$. We think of $[k]$ as colors.

A function $c : V(G) \to [k]$ is a proper coloring of G with at most k colors, if for each $i \in [k]$ the set $c^{-1}(i)$ is an independent set in G.

We denote by $\chi(G; k)$ the number of proper colorings of G with k colors.

Birkhoff (1912) showed that $\chi(G; k)$ is a polynomial in $\mathbb{Z}[k]$. Furthermore, he showed that for the edgeless graph with n vertices, $E_n = ([n], \emptyset)$ we have that $\chi(E_n; k) = k^n$ and

$$\chi(G_{/e}; k) = \chi(G; k) + \chi(G_{\backslash e}; k)$$

where $e \in E(G)$ and $G_{/e}$ is obtained by deleting the edge e and $G_{\backslash e}$ is obtained by contracting the edge e.

This shows that $\chi(G; k)$ is a polynomial counting colorings and that $\chi(G; k)$

has a recursive definition,

and can be **extended** to a polynomial $\chi(G; X) \in \mathbb{R}[X]$.

Harary's $\mathcal{P}$-colorings

Let $\mathcal{P}$ be any graph property and $k \in \mathbb{N}$ with $[k] = \{1, 2, \dots k\}$.

A $\mathcal{P}$-vertex coloring of a graph $G = (V(G), E(G)$ with at most k colors is a function f

$$f : V(G) \to [k]$$

such that for each $i \in [k]$, the set $f^{-1}(i)$ induces a graph $G[f^{-1}(i)] \in \mathcal{P}$.

- If $\mathcal{P}$ is the class of edgeless graphs, we get the proper vertex colorings.
- If $\mathcal{P}$ is the class of connected graphs, we get the convex vertex colorings.

Counting $\mathcal{P}$-colorings

We denote by

$$\chi_{\mathcal{P}}(G;k)$$

the number of distinct $\mathcal{P}$-colorings of G with at most k colors.

Theorem 1 (JAM and Zilber, 2005)

For every graph G, the counting function
$\chi_{\mathcal{P}}(G;k)$ is a polynomial in $\mathbb{Z}[k]$,

$\chi_{\mathcal{P}}(G;k)$ has a canonical extension
to a polynomial over the real or complex numbers,
which we denote by $\chi_{\mathcal{P}}(G;X)$.

Definability, I

Let $\mathcal{L}$ be a fragment of Second Order Logic SOL.

FOL is First Order Logic.
MSOL_{graph} is Monadic Second Order Logic in the language of graphs,
$\mathrm{MSOL}_{hypergraph}$ is Monadic Second Order Logic in the language of hypergraphs, and similar for CMSOL with modular counting.

- A Harary polynomial is $\mathcal{L}$-definable, or an $\mathcal{L}$-Harary polynomial, if $\mathcal{P}$ is $\mathcal{L}$-definable.

- A graph polynomial is an $\mathcal{L}$-polynomial if it is of the form

$$P(G;\bar{X}) = \sum_{R_1 : G \models \phi(R_1)} \cdots \sum_{R_k : G \models \phi(R_1,\ldots,R_k)} \prod_{S_1 : G \models \phi(\bar{R},S_1)} X_1 \cdots \prod_{S_\ell : G \models \phi(\bar{R},S_1,\ldots,S_\ell)} X_\ell$$

Definability, II

Theorem 2 (Kotek and JAM, 2008/12)

(i) *Every* SOL*-Harary polynomial is an* SOL*-polynomial.*

(ii) *The chromatic polynomial* $\chi(G; X)$ *is a* FOL*-Harary polynomials which is not a* CMSOL_{graph}*-polynomial, but is a* $\mathrm{CMSOL}_{hypergraph}$*-polynomial on ordered hypergraphs (in an ordered invariant way).*

(iii) *There are* FOL*-Harary polynomials which are not* $\mathrm{CMSOL}_{hypergraph}$*-polynomials.*

Which mathematical properties

are shared

between $\chi(G;X)$ and $\chi_{\mathcal{P}}(G;X)$?

Multiplicativity

Let $G \sqcup H$ denote the disjoint union of the graphs G and H.

- The chromatic polynomial is multiplicative, i.e.,

$$\chi(G \sqcup H; X) = \chi(G; X) \cdot \chi(H; X)$$

A graph property $\mathcal{P}$ is hereditary if for all $G \in \mathcal{P}$ and H an induced subgraph $H \subset_{ind} G$ we have $H \in \mathcal{P}$.

A graph property $\mathcal{P}$ is additive if for all $G, H \in \mathcal{P}$ we have $G \sqcup H \in \mathcal{P}$.

Proposition 3

Assume a graph property $\mathcal{P}$ is hereditary and additive.

Then $\chi_{\mathcal{P}}(G; X)$ is multiplicative, i.e.,

$$\chi_{\mathcal{P}}(G \sqcup H; X) = \chi_{\mathcal{P}}(G; X) \cdot \chi_{\mathcal{P}}(H; X)$$

Deletion/contraction relations, I

Let G be an undirected graph, possibly with multiple edges and loops. For $e = \{u, v\} \in E(G)$ we denote by

- G_{-e} the graph with
$$V(G_{-e}) = V(G) \text{ and } E(G_{-e}) = E(G) - \{e\}.$$

- $G_{/e}$ the graph with
$$V(G_{/e}) = (V(G) - \{u, v\}) \cup \{w\}$$
where $w \notin V(G)$, and
$$E(G_{/e}) = (E(G) - \{\{a, u\} : a \in V(G)\} - \{\{a, v\} : a \in V(G)\}) \\ \cup \{\{a, w\} : \{a, u\} \in E(G) \text{ or } \{a, v\} \in E(G)\})$$

Deletion/contraction relations, II

The chromatic polynomial satisfies the following deletion/contraction relation:

$$\chi(G;X) = \chi(G_{-e};X) - \chi(G_{/e};X)$$

Two graphs are similar if the have the same number $n(G)$ of vertices, $m(G)$ of edges, and $k(G)$ of connected components.

A graph polynomial $\chi_{\mathcal{P}}(G;X)$ is a **chromatic invariant**
(aka a Tutte-Grothendieck invariant) **if**

$$\chi_{\mathcal{P}}(G;X) = \begin{cases} \alpha^{n(G)} & \textbf{if } G \textbf{ has no edges} \\ \beta \cdot \chi_{\mathcal{P}}(G_{-e};X) & \textbf{if } e \textbf{ is a loop} \\ \gamma \cdot \chi_{\mathcal{P}}(G_{/e};X) & \textbf{if } e \textbf{ is a bridge} \\ \lambda \cdot \chi_{\mathcal{P}}(G_{-e};X) + \mu \cdot \chi_{\mathcal{P}}(G_{/e};X) & \textbf{otherwise} \end{cases}$$

Distinctive power of graph parameters

For graph G we denote by $n(G)$ the number of vertices, $m(G)$ the number of edges and $k(G)$ the number of connected components.

- Two graphs G_1, G_2 are similar if $n(G_1) = n(G_2)$, $m(G_1) = m(G_2)$ and $k(G_1) = k(G_2)$.
- Let f, g be two graph parameters. f is at least as distinctive as g (on similar graphs) if for all (similar) graphs G_1, G_2 we have that $f(G_1) = f(G_2)$ implies $g(G_1) = g(G_2)$.

  We write $g \leq_{d.p} f$, respectively $f \leq_{s.d.p} g$.
- f and g have the same distinctive power (are d.p.-equivalent) if for all graphs G_1, G_2

$$f(G_1) = f(G_2) \text{ iff } g(G_1) = g(G_2).$$

- They have the same distinctive power up to similarity (are s.d.p.-equivalent) if for all similar graphs G_1, G_2

$$f(G_1) = f(G_2) \text{ iff } g(G_1) = g(G_2).$$

The most general chromatic invariant

Theorem 4 (T. Brylawski)
There is chromatic invariant $U(G; X_1, \ldots, X_5)$ such that

- *for every other chromatic invariant f we have $f \leq_{s.d.p} U$.*
- *The Tutte polynomial $T(G; X, Y)$ is s.d.p-equivalent to U.*

Two graphs G_1 and G_2 are T-equivalent if $T(G_1, X, Y) = T(G_2, X, Y)$.

Theorem 5 (T. Brylawski)
There infinitely many pairs of non-isomorphic similar graphs G_1 and G_2 which are T-equivalent.

Are $\mathcal{P}$-colorings chromatic invariants?

Observation: If two graphs G_1 and G_2 are non-isomorphic, there is graph property $\mathcal{P}$ such $G_1 \in \mathcal{P}$ and $G_2 \notin \mathcal{P}$.

In fact $\mathcal{P}$ is even definable in first order logic FOL.

Theorem 6 (T. Kotek, JAM, E.V. Ravve)

(i) There are infinitely many FOL*-definable properties* $\mathcal{P}$ *such that* $\chi_{\mathcal{P}}(G; X)$ *is not a chromatic invariant.*

(ii) There are infinitely many FOL*-definable properties* $\mathcal{P}$ *such that the graph polynomials* $\chi_{\mathcal{P}}(G; X)$ *are mutually* **s.d.p.-incomparable***,
and therefore also* **d.p.-incomparable***.*

However, in (i) $\mathcal{P}$ is not explicitly given, and in (ii) $\mathcal{P}$ contains, up to isomorphism, just a single structure.

Can we do better?

H-free graphs

Let H be a simple graph. A graph G is H-free if it does not contain H as an induced subgraph.

We denote by $Fr(H)$ the class of H-free graphs and note that $Fr(H)$ is FOL-definable, hereditary, and for H connected, also monotone.

We look at the FOL-Harary polynomial $\chi_{Fr(H)}(G;X)$.

Theorem 7 (V. Rakita and JAM, 2017)

- *If H is connected, $\chi_{Fr(H)}(G;X)$ is multiplicative.*
- *If H is not a complete graph, $\chi_{Fr(H)}(G;X)$ is not a chromatic invariant.*
- *There infinitely many pairs of graphs H, H' such that $\chi_{Fr(H)}(G;X)$ and $\chi_{Fr(H')}(G;X)$ are s.d.p.-incomparable.*

Some open problems

- Which graph polynomials $\chi_{\mathcal{P}}(G; X)$ are chromatic invariants?

- Is the chromatic polynomial the only graph polynomial $\chi_{\mathcal{P}}(G; X)$ which is a chromatic invariant?

- Find other infinite families $\mathcal{X}$ of graph properties such that the polynomials $\chi_{\mathcal{P}}(G; X), \mathcal{P} \in \mathcal{X}$ are mutually s.d.p.-incomparable!

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Part II (i):

The complexity of evaluating the chromatic polynomial $\chi(G; X)$

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The complexity of the chromatic polynomial, I

Theorem:

- $\chi(G, 0)$, $\chi(G, 1)$ and $\chi(G, 2)$ are P-time computable (Folklore)
- $\chi(G, 3)$ is $\sharp\mathbf{P}$-complete (Valiant 1979).
- $\chi(G, -1)$ is $\sharp\mathbf{P}$-complete (Linial 1986).

Question:

What is the complexity of computing $\chi(G, X)$ for

$X = X_0 \in \mathbb{Q}$

or even for

$X = X_0 \in \mathbb{C}$?

The complexity of the chromatic polynomial, II
Linial's Trick

Let $G_1 \bowtie G_2$ denote the join of two graphs.

We observe that

$$\chi(G \bowtie K_n, X) = (X)^{\underline{n}} \cdot \chi(G, X - n) \qquad (\star)$$

Hence we get

(i) $\chi(G \bowtie K_1, 4) = 4 \cdot \chi(G, 3)$

(ii) $\chi(G \bowtie K_n, 3 + n) = (n + 3)^{\underline{n}} \cdot \chi(G, 3)$
hence for $n \in \mathbb{N}$ with $n \geq 3$ it is $\sharp$**P-complete**.

The complexity of the chromatic polynomial, III

We have a **Dichotomy Theorem** for the evaluations of $\chi(-, X)$:

(i) EASY$(\chi) = \{a \in \mathbb{C} : \chi(-, a) \in \mathbf{P}\} = \{0, 1, 2\}$

Moreover, in $\mathbb{C}$ this is a quasi-algebraic set (a finite boolean combination of algebraic sets) of dimension 0.

(ii) HARD$(\chi) = \{a \in \mathbb{C} : \chi(-, a)$ is $\chi(-; 3)$-hard$\} = \mathbb{C} - \{0, 1, 2\}$

More precisely, they are at least as difficult as $\chi(-, 3)$ via algebraic reductions.

This is a quasi-algebraic set of dimension 1.

Dichotomy: there are no intermediate complexities, although, if $\mathbf{P} \neq \mathbf{P}^{\sharp\mathbf{P}}$ there would be plenty of complexity levels in between.

In the sequel we speak of the **Difficult Point Dichotomy (DPD)**.

How typical is DPD?

Skip computational models

Part II:(ii)

Issues with the computational model

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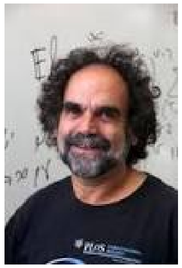
Turing complexity: $\mathbf{P}$ vs $\sharp\mathbf{P}$


N. Linial


L. Valiant

Let K be a fixed Turing computable subfield $K \subseteq \mathbb{C}$.

(i) EASY$(\chi) = \{a \in K : \chi(-,a) \in \mathbf{P}\} = \{0,1,2\}$

(ii) HARD$(\chi) = \{a \in K : \chi(-,a)$ is $\sharp\mathbf{P}$-hard$\} = K - \{0,1,2\}$

Problems:

(i) For $a \in \mathbb{C} - \mathbb{N}$ the graph parameter $\chi(-;a)$ is **not well defined** in the Turing model, and definitely is not in $\sharp\mathbf{P}$.

(ii) The precise statement **depends on the choice of K** and its presentation as a Turing computable field.

Computing over the reals: $\mathbf{P}_{\mathbb{R}}$ vs $\sharp\mathbf{P}_{\mathbb{R}}$

L. Blum, F. Cucker, M. Shub, S. Smale

K. Meer

Here we use **register machines over a ring** $\mathcal{R}$, for our discussion $\mathcal{R} = \mathbb{R}$ or $\mathcal{R} = \mathbb{C}$. $\mathbf{P}_{\mathbb{R}}$ and $\mathbf{NP}_{\mathbb{R}}$ are well defined, and there are $\mathbf{P}_{\mathbb{R}}$-complete problems.

Counting classes are defined by counting solutions of systems of equations, but $\sharp\mathbf{P}_{\mathbb{R}}$ does not differentiate between the ways one could have infinitely many solutions.

(i) EASY$(\chi) = \{a \in K : \chi(-, a) \in \mathbf{P}_{\mathbb{R}}\} = \{0, 1, 2\}$

(ii) HARD$(\chi) = \{a \in K : \chi(-, a)$ is $\chi(-; 3) -$ hard$\} = \mathbb{R} - \{0, 1, 2\}$

Problems:

(i) What do we know about $\chi(-; 3) -$ hard-problems the BSS-model of computation?

(ii) Are they all in $\sharp\mathbf{P}_{\mathbb{R}}$ or in $\sharp\mathbf{P}_{\mathbb{C}}$?

Skip Metafinite

Metafinite Model Theory

 E. Graedel  Y. Gurevich

A metafinite W-structure $\mathfrak{A}$ over $\mathbb{R}$ is given by an underling set finite set $[n_A]$ of n_A elements, together with a finite set of functions $W = \{f_1, \ldots, f_s\}$ of arity $\rho(i)$

$$f_i^A : [n]^{\rho(i)} \to \mathbb{R}$$

.

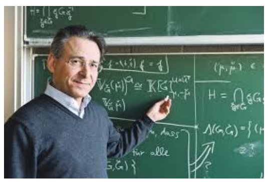
Two metafinite W structures $\mathfrak{A}, \mathfrak{B}$ are W-isomorphic if there is a bijection $\alpha : n_A \to n_B$ such that for each $\rho(i)$-tuple $\bar{a} \in [n_A]^{\rho(i)}$ we have that

$$f_i^B(\alpha(\bar{a})) = f_i^A(\alpha(\bar{a})).$$

An $\mathbb{R}$-parameter of a metafinite W-structures parametrized by s-tuples $\bar{a} \in \mathbb{R}^s$ is a function $P(\mathfrak{A}; \bar{a})$ into $\mathbb{R}$ which is **invariant** under W-isomorphisms.

The chromatic polynomial of a graph is an $\mathbb{R}$ parameter of graphs.

$\sharp\mathbf{P}_{\mathbb{R}}$ vs $\sharp\mathbf{P}_{\mathbb{C}}$

 P. Buergisser

 F. Cucker

- (K. Meer) $\mathrm{FP}_{\mathbb{R}} \subseteq \sharp\mathbf{P}_{\mathbb{R}} \subseteq \mathrm{FP}_{\mathbb{R}}^{\sharp\mathbf{P}_{\mathbb{R}}} \subseteq EXP_{\mathbb{R}}$.
- $\mathrm{FP}_{\mathbb{C}} \subseteq \sharp\mathbf{P}_{\mathbb{C}} \subseteq \mathrm{FP}_{\mathbb{C}}^{\sharp\mathbf{P}_{\mathbb{C}}} \subseteq EXP_{\mathbb{C}}$.
- (Buergisser, Cucker) $\sharp\mathbf{P}_{\mathbb{R}}$, $\mathrm{FP}_{\mathbb{R}}^{\sharp\P_{\mathbb{R}}}$ and $\sharp\mathbf{P}_{\mathbb{C}}$ have complete problems for polynomial time Turing reductions.
- However, the fine structure of $\sharp\mathbf{P}_{\mathbb{R}}$ and $\sharp\mathbf{P}_{\mathbb{C}}$ is not well understood.

Counting in $\sharp \mathbf{P}_{\mathbb{C}}$

Let G be a graphs of order n with $V(G) = [n]$.

Let $F(G;k)$ be the following set of equations:

$$x_i^k - 1 = 0, i \in V(G)$$

and

$$\sum_{d=0}^{k-1} x_i^{k-1-d} x_j^d = 0, (i,j) \in E(G)$$

and $\sharp F(G;k)$ the number of its solution over $\mathbb{C}$.

Theorem: (D. Bayer) Let G be a graph and $k \in \mathbb{N}$.

- G is k-colorable iff the system of equations $F(G,k)$ has a solution.
- Furthermore, $\chi(G;k) \cdot k! = \sharp F(G;k)$.

What is the complexity spectrum of $\chi(-; a)$

Let $\mathcal{F}$ be a recursive infinite field extending $\mathbb{Q}$,
with polynomial time Turing-computable basic operations.

Upper bounds for $\chi(-; a)$

$\chi(-; a)$	Turing $\mathcal{F}$	Turing $\mathbb{R}$	Turing $\mathbb{C}$	BSS $\mathcal{F}$	BSS $\mathbb{R}$	BSS $\mathbb{C}$
$a = 0, 1, 2$	$\mathbf{P}$	$\mathbf{P}$	$\mathbf{P}$	$\mathbf{P}_{\mathcal{F}}$	$\mathbf{P}_{\mathbb{R}}$	$\mathbf{P}_{\mathbb{C}}$
$a \in \mathbb{N} - \{0, 1, 2\}$	$\sharp\mathbf{P}$	$\sharp\mathbf{P}$	$\sharp\mathbf{P}$	$\mathsf{EXP}_{\mathcal{F}}$	$\mathsf{EXP}_{\mathbb{R}}$	$\sharp\mathbf{P}_{\mathbb{C}}$
$a \notin \mathbb{N}$	$\sharp\mathbf{P}$	undefined	undefined	$\mathsf{EXP}_{\mathcal{F}}$	$\mathsf{EXP}_{\mathbb{R}}$	$\mathsf{FP}_{\mathbb{C}}^{\sharp\mathbf{P}_{\mathbb{C}}}$

Lower bounds for $\chi(-; a)$

$\chi(-; a)$	Turing $\mathcal{F}$	Turing $\mathbb{R}$	Turing $\mathbb{C}$	BSS $\mathcal{F}$	BSS $\mathbb{R}$	BSS $\mathbb{C}$
$a = 0, 1, 2$	$\mathbf{P}$	$\mathbf{P}$	$\mathbf{P}$	$\mathbf{P}_{\mathcal{F}}$	$\mathbf{P}_{\mathbb{R}}$	$\mathbf{P}_{\mathbb{C}}$
$a \in \mathbb{N} - \{0, 1, 2\}$	$\sharp\mathbf{P}$	$\sharp\mathbf{P}$	$\sharp\mathbf{P}$	$\mathbf{P}_{\mathcal{F}}$	$\mathbf{P}_{\mathbb{R}}$	$\mathbf{P}_{\mathbb{C}}$
$a \notin \mathbb{N}$	$\sharp\mathbf{P}$	undefined	undefined	$\mathbf{P}_{\mathcal{F}}$	$\mathbf{P}_{\mathbb{R}}$	$\mathbf{P}_{\mathbb{C}}$

An unsatisfactory situation

- The statement that using $\chi(-;a)$ as an oracle with $a \notin \mathbb{N}$ allows us to compute $\chi(-;b)$ for $b \in \mathbb{N}$ in polynomial time in the BSS model of computation does not imply that computing $\chi(-;a)$ is hard in the BSS model of computation.

- The mixing of two computational models is really **meaningless**.

- The traditional excuse that dealing with a fixed computable extension $\mathcal{F}$ of $\mathbb{Q}$ makes a statement about the complexity in the Turing model of computation in $\mathcal{F}$ for all $a \in \mathcal{F}$, but does not address the issue for $a \in \mathbb{R} - \mathcal{F}$.

- The same remarks apply also to other graph polynomial mentioned in the sequel.

This was the topic of my talk at the workshop last year

The Classification Program of Counting Complexity

Go to thanks

Part II (iii):

The Difficult Point Properties (DPP)

and the Difficult Point Dichotomy

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Difficult Point Property, I

Given a graph polynomial $P(G, \bar{X})$ in n indeterminates $X_1, \ldots, X_n$

we are interested in the set HARD(P).

(i) We say that P has the **weak difficult point property (WDPP)** if HARD(P) $\neq \emptyset$ then there is a quasi-algebraic subset $D \subset \mathbb{C}^n$ of co-dimension $\leq n-1$ such that $\mathbb{C}^n - D \subseteq$ HARD(P).

(ii) We say that P has the **strong difficult point property (SDPP)** if HARD(P) $\neq \emptyset$ then there is a quasi-algebraic subset $D \subset \mathbb{C}^n$ of co-dimension $\leq n-1$ such that $\mathbb{C}^n - D =$ HARD(P) $\neq \emptyset$ and $D =$ EASY(P).

In both cases EASY(P) is of dimension $\leq n-1$, and for almost all points $\bar{a} \in \mathbb{C}^n$ the evaluation of $P(-, \bar{a})$ is $\sharp\mathbf{P}$-hard.

$\chi(G; \lambda)$ has the SDPP.

Difficult Point Property, II

We compare WDPP and SDPP to Dichtomy Properties.

(i) We say that P has the **dichotomy property (DiP)** if
$\mathrm{HARD}(P) \cup \mathrm{EASY}(P) = \mathbb{C}^n$.
Clearly, if $\mathbf{P}_{\mathbb{C}} \neq \mathbf{NP}_{\mathbb{C}}$, $\mathrm{HARD}(P) \cap \mathrm{EASY}(P) = \emptyset$.

(ii) WDPP is **not** a dichtomy property, but **SDPP a dichotomy property**.

(iii) The two versions of DPP have a quantitative aspect:

$\mathrm{EASY}(P)$ **is small**.

Graph polynomials with the DPP, I

SDPP: The Tutte polynomial (our paradigma).

SDPP: the cover polynomial $C(G, x, y)$ introduced
by Chung and Graham (1995)
by Bläser, Dell 2007, Bläser, Dell, Fouz 2011

SDPP: the bivariate matching polynomial for multigraphs,
by Averbouch and JAM, 2007

WDPP: the Bollobás-Riordan polynomial, generalizing the Tutte polynomial and introduced by Bollobás and Riordan (1999),
by Bläser, Dell and JAM 2008, 2010.

WDPP: the interlace polynomial (aka Martin polynomial) introduced by Martin (1977) and independently by Arratia, Bollobás and Sorkin (2000),
by Bläser and Hoffmann, 2007, 2008

Skip partition functions

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Partition functions as graph polynomials

- Let $A \in \mathbb{C}^{n \times n}$ a symmetric and G be a graph. Let
$$Z_A(G) = \sum_{\sigma : V(G) \to [n]} \prod_{(v,w) \in E(G)} A_{\sigma(v),\sigma(w)}$$
Z_A is called a partition function.
- Let $\mathbf{X}$ be the matrix $(X_{i,j})_{i,j \leq n}$ of indeterminates.
Then $Z_{\mathbf{X}}$ is a graph polynomial in n^2 indeterminates,
Z_A is an evaluation of $Z_{\mathbf{X}}$, and $Z_{\mathbf{X}}$ is MSOL-definable.

Partition functions have the SDPP

- J. Cai, X. Chen and P. Lu (2010),
  building on A. Bulatov and M. Grohe (2005),
  proved a dichotomy theorem for $Z_{\mathbf{X}}$ where $\mathcal{R} = \mathbb{C}$.

- Analyzing their proofs reveals:
  $Z_{\mathbf{X}}$ satisfies the SDPP for $\mathcal{R} = \mathbb{C}$.

- There are various generalizations of this to Hermitian matrices,
  M. Thurley (2009),
  and beyond.

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Which graph polynomials of the form

$$\chi_{\mathcal{P}}(G; X)$$

have some form of a dychotomy property ?

Part III: On the complexity

of **generalized** chromatic polynomials.

The univariate case

GHKMN, Advances in Applied Mathematics,

http://dx.doi.org/10.1016/j.aam.2017.04.005

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The complexity spectrum of a univariate graph polynomial

We shall look at other univariate graph polynomials $Q(G; X)$

- EASY$(Q) = \{a \in \mathbb{C} : Q(-, a) \in \mathbf{P}_{\mathbb{C}}\}$
- For a graph polynomial $Q(G; X)$ and $a, b \in \mathbb{C}$ let $a <^{P}_{poly} b$ if the graph parameters $Q(, -a)$ is algebraically reducible to $Q(-, b)$ in polynomial time.
- HARD$(Q) = \{a \in \mathbb{C} : Q(-, a_0)$ is $\sharp\mathbf{P}$-hard and $Q(-, a_0) <^{P}_{poly} Q(-, a)\}$ for some a_0.

  Here $\sharp\mathbf{P}$-hard is in the Turing model and $<_P$ is in BSS.
- For which graph polynomials do we have a dichotomy?

What is the structure of the partial order defined by $a <^{P}_{poly} b$ for various univariate graph polynomials $P(G; X)$?

Easy evaluations

Some graph polynomials are always easy to evaluate:

- The characteristic polynomial and the Laplacian polynomial, because they are defined as the characteristic polynomial of the adjacency, resp. the indicidence matrix of the graph, which is a determinant.

- $\chi_{connected}(G; k)$ is the number of vertex colorings with at most k colors such that neighboring vertices have the same color.

  This is clearly the polynomial $k^{\kappa}(G)$ where $\kappa(G)$ is the number of connected components of G.

This gives us that **SDPP** holds in a trivial way

(as there are no difficult points).

Proper edge colorings $\chi_{edge}(G; X)$, I

- A proper edge coloring f of a graph G,
  $f : E(G) \rightarrow [k]$ with at most k colors,
  is an edge coloring where no two neighboring edges have the same color.

- f is a proper edge coloring of G iff f is is a proper vertex coloring of the line graph $L(G)$.

- Therefore, the number $\chi_{edge}(G; k)$ of proper edge colorings of G with at most k colors is a polynomial in k.

Proper edge colorings $\chi_{edge}(G; X)$, II

Surprisingly, the complexity of counting proper edge colorings was proven $\sharp\mathbf{P}$-hard only recently:

Theorem: (J. Y. Cai, H. Guo, T. Williams, 2014):

- $\sharp$-EdgeColoring is $\sharp\mathbf{P}$-hard over planar r-regular graphs for all $k \geq r \geq 3$.
- It is trivially tractable when $k \geq r \geq 3$ does not hold.

J. Y. Cai, H. Guo, T. Williams
The complexity of counting edge colorings and a dichotomy
for some higher domain Holant problems,
FOCS 2014 (full paper on arXiv `http://arxiv.org/pdf/1404.4020.pdf`, 75 pages)

Problem: Find an elementary proof of the complexity result.

Proper edge colorings $\chi_{edge}(G; X)$, III

What about evaluation of $\chi_{edge}(G; X)$ for $X = a, a \in \mathbb{C} - \mathbb{N}$?

We could not find a variant of Linial's Trick for $\chi_{edge}(G; X)$.

Equivalently, what about the complexity of the chromatic polynomial restricted to line graphs?

Line graphs can be characterized using 9 forbidden induced subgraphs.

The problem is wide open.

There are two problems to be solved:

- Show that for at least some value $X = a_0$ the evaluation of $P(G; X)$ is hard.
- Show that for most values $X = a$ the evaluation is at least as hard as for $X = a_0$.

Both problems may turn out to be new challenges!

Convex colorings

Joint work with A. Goodall and S. Noble

A convex (vertex) coloring with k colors is **convex** if every monochromatic set of vertices induces a connected graph.

Theorem: The problem of counting the number of colorings of the vertices of a graph with at most two colors, such that the color classes induce connected subgraphs is $\sharp\mathbf{P}$-complete.

A. Goodall and S. Noble, 2008 (http://arxiv.org/pdf/1404.4020.pdf)

I had posed this as an open problem at L. Lovasz' 60th birthday conference, after trying to prove it and discussing it with Peter Winkler.

Here the reduction is simple:

$$\chi_{convex}(G \sqcup K_1; X+1) = X \cdot \chi_{convex}(G; X)$$

Computing $\chi_{convex}(G; 0)$ and $\chi_{convex}(G; 1)$ is easy.

This gives us that **SDPP** holds.

Complete and harmonious colorings

Joint work with T. Kotek

A coloring is

- **complete** if **every pair of colors occurs along some edge**.
- **harmonious** if **every pair of colors occurs at most once along some edge**.
- That $\chi_{harm}(G;k)$ is a polynomial in k, was shown by B. Zilber and JAM.
- $\chi_{complete}(G;k)$ is not a polynomial in k.

The exact complexity for fixed k seemingly is open......

Harmonious colorings, continued.

Proposition: For every $k \in \mathbb{N}$ $\chi_{harm}(-;k)$ is easy to compute for $k \in \mathbb{N}$, because there are only finitely many graphs without isolated vertices which admit a harmonious coloring with k-colors.

However, this is not uniform: For each k a different polynomial time Turing machine is used.

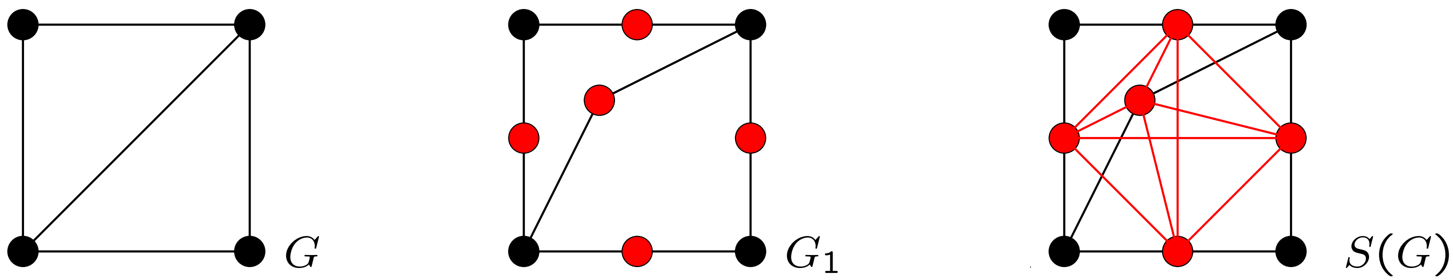
Theorem: For each $x \in \mathbb{C} - \mathbb{N}$ the evaluation of $\chi_{harm}(G;x)$ is $\sharp\mathbf{P}$-hard.

This gives us that **SDPP does not** hold ($\mathbb{N}$ is not semi-algebraic) for **harmonious** colorings.

However, **DPD does hold**.

Harmonious colorings, proof, I

We show that for each $x \in \mathbb{C} - \mathbb{N}$ the evaluation of $\chi_{harm}(G; x)$ is $\sharp\mathbf{P}$-hard.



We add a **red** vertex on each edge of G (making two **black** edges out of it) and then add **red** edges such that the **red** vertices form a clique.

First we note that

$$\chi_{har}(S(G); k + e) = \chi(G; k) \cdot \binom{k + e}{e} e!$$

where $e = |E(G)|$ and $\chi(G; k)$ is the chromatic polynomial.

Harmonious colorings, proof, II

- Now for $k = a$ we have
$$\frac{\chi_{har}(S(G); a)}{\binom{a}{e} e!} = \chi(G; a - e)$$

- It remains to be shown that
$$\chi(G; a - e)$$
is is $\sharp\mathbf{P}$-hard for every $a \in \mathbb{C} - \mathbb{N}$.

- We use Linial's Trick:

  Let $v = |V(G)|$ and $|E(G \bowtie K_1)| = e + v$:
$$\chi(G \bowtie K_1; a - (e + v) + 1) = (a - (e + v) + 1) \cdot \chi(G, a - (e + v))$$
  Which can be used for every $a \in \mathbb{C} - \mathbb{N}$.

$mcc(t)$-colorings

Let $f : V \to [k]$ be an vertex-coloring and $t \in \mathbb{N}$.

f is an $mcc(t)$-coloring of G with k colors, if all the connected components of a monochromatic set have size at most t.

N. Alon, G. Ding, B. Oporowski, and D. Vertigan. Partitioning into graphs with only small components. Journal of Combinatorial Theory, Series B, 87:231–243, 2003.

Theorem: (JAM, B. Zilber) Counting the number of $mcc(t)$-colorings with at most k colors is a polynomial in k but not in t, and is denoted by $\chi_{mcc(t)}(G; X)$.

$mcc(t)$-colorings

Joint work with Miki Hermann

Theorem: For every $t \in \mathbb{N}^+$, computing $\chi_{mcc(t)}(G; 2)$ is $\sharp\mathbf{P}$-hard.

Proof: Reduction to $\sharp$NAEkSAT.

$\sharp$NAEkSAT is $\sharp\mathbf{P}$-complete for $k \geq 3$ by

Creignou, Nadia, and Miki Hermann. "Complexity of generalized satisfiability counting problems." Information and Computation 125.1 (1996): 1-12.

We **failed** to see how to use a version of Linial's Trick.

We: A. Goodall, M. Hermann, T. Kotek, JAM and S. Noble.

Open Problem: What is the full complexity spectrum of $\chi_{mcc(t)}(G; 2)$ for $t \geq 2$?

$DU(H)$-colorings, I

In our attempt to determine the complexity spectrum of $\chi_{mcc(t)}(G;X)$ we studied $DU(H)$-colorings.

Here H is a connected graph and an edge coloring $f : V(G) \rightarrow [k]$ is an $DU(H)$-coloring if each color set induces a disjoint union of copies of H.

- If $H = K_1$ these are the proper colorings.
- $\chi_{DU(H)}(G;k)$ is a polynomial in k.
- Furthermore, every $DU(K_t)$ coloring is also a $mcc(t)$-coloring, and

$$\chi_{DU(K_t)}(G;k) \leq \chi_{mcc(t)}(G;k).$$

Theorem: For every $t \in \mathbb{N}^+$ evaluating $\chi_{DU(K_t)}(G;2)$ is $\sharp\mathbf{P}$-complete.

What remains is a version of Linial's Trick.

$DU(H)$-colorings, II

Let $v \in V(H)$. We define $\square_{H,v}(G)$ to be the graph with vertex set $V(G) \sqcup V(H)$, and edge set $E(G) \sqcup E(H) \sqcup V(G) \times \{v\}$.

Let H be a connected graph.

(i) $\chi_{DU(H)}(\square_{H,v}(G); k) = k \cdot \chi_{DU(H)}(G; k-1)$.

(ii) For every $a, b \in \mathbb{N}$ and $b > a$, $\chi_{DU(H)}(G; a)$ is polynomial time reducible to $\chi_{DU(H)}(G; b)$.

(iii) For every $a_0 \in \mathbf{F} - \mathbb{N}$, computing the coefficients of $\chi_{DU(H)}(G; X)$ is polynomial time reducible to $\chi_{DU(H)}(G; a_0)$.

Show proof

Skip proof

Proof for $DU(H)$-colorings

Proof:
(i) All the vertices of $V(H)$ have to be colored by the same color but differently from the vertices in $V(G)$.
(ii) Apply (i) $b - a$ many times.
(iii) Let $G_0 = G$, $G_{i+1} = \square_{H,v}(G_i)$. Using $\chi_{DU(H)}(-; a_0)$ we can compute $\chi_{DU(H)}(G_i; a_0)$ for sufficiently many i's and then use Lagrange Interpolation to compute the coefficients of $\chi_{DU(H)}(G; X)$. Q.E.D.

More variations on coloring, I

More coloring polynomials in $\mathbb{Z}[k]$:

* **injective:** f is injectiv on the neighborhood of every vertex.
  G. Hahn and J. Kratochvil and J. Siran and D. Sotteau, On the injective chromatic number of graphs, Discrete mathematics, 256.1-2, (2002), 179-192.

* **path-rainbow:** Let $f : E \to [k]$ be an edge-coloring. f is **path-rainbow** if between any two vertices $u, v \in V$ there as a path where all the edges have different colors.

  Rainbow colorings of various kinds arise in computational biology

  Rainbow connection in graphs, G. Chartrand and G.L. Johns and K. McKeon A and P. Zhang, Mathematica Bohemica, 133.1, (2008), 85-98.

More variations on coloring, II

Let $\mathcal{P}$ be any graphs property and let $n \in \mathbb{N}$.

We can define coloring functions $f : V \to [k]$ by requiring that the union of any n color classes induces a graph in $\mathcal{P}$.

- For $n = 1$ and $\mathcal{P}$ the empty graphs $G = (V, \emptyset)$ we get the proper colorings.
- For $n = 1$ and $\mathcal{P}$ the connected graphs we get the convex colorings.
- For $n = 1$ and $\mathcal{P}$ the graphs which are disjoint unions of graphs of size at most t, we get the mcc_t-colorings.
- For $n = 2$ and $\mathcal{P}$ the acyclic graphs we get the **acyclic colorings**, introduced in: B. Grunbaum, Acyclic colorings of planar graphs, Israel J. Math. 14 (1973), 390-412 and further studied in N.Alon , C. Mcdiarmid, B. Reed, Acyclic coloring of graphs, Random Structures & Algorithms 2.3 (1991) 277-288.

Theorem: Let $\chi_{\mathcal{P},n}(G, k)$ be the number of colorings of G with k colors such that the union of any n color classes induces a graph in $\mathcal{P}$.

Then $\chi_{\mathcal{P},n}(G, k)$ is a polynomial in k.

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More univariate graph polynomials, III

$\mathbf{CP}$-coloring	$\mathcal{P}_1$	$\mathcal{P}_2$
trivial	all graphs	all graphs
proper	edgeless graphs	all graphs
acyclic	edgeless graphs	forests
convex	connected graphs	all graphs
harmonious	edgeless graphs	at most one edge
mcc$_t$	conn. cpts size $\leq t$	all graphs
$DU(H)$	disjoint union of $\cong H$	all graphs
t-imp	max. degree t	all graphs
co-coloring	clique or edgeless	all graphs
$\mathcal{AH}$-coloring	$\mathcal{AH}$	all graphs

$\mathcal{P}_1$-colorings where the union of any two color classes is in $\mathcal{P}_2$. In the last line $\mathcal{P}_1$ is an additive induced hereditary property (closed under taking induced subgraphs and disjoint unions).

A good test problem: H-free colorings, I

We look at the generalized chromatic polynomial $\chi_{H-free}(G;k)$, which, for $k \in \mathbb{N}$ counts the number of H-free colorings of G.

- For $H = K_2$, $\chi_{H-free}(G;k) = \chi(G;k)$, and we have the SDPP.
- For $H = K_3$, $\chi_{H-free}(G;k)$ counts the triangle free-colorings.

A good test problem: H-free colorings, II

- From [ABCM98] it follows that $\chi_{H-free}(G;k)$ is #P-hard for every $k \geq 3$ and H of size at least 2.

  D. Achlioptas, J. Brown, D. Corneil, and M. Molloy. The existence of uniquely -G colourable graphs. *Discrete Mathematics*, 179(1-3):1–11, 1998.

- In [Achlioptas97] it is shown that computing $\chi_{H-free}(G;2)$ is NP-hard for every H of size at most 2.

  D. Achlioptas. The complexity of G-free colourability. *DMATH: Discrete Mathematics*, 165, 1997.

- **Characterize H for which $\chi_{H-free}(G;k)$ satisfies the SDPP (WDPP).**

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Conclusions:

Challenges and summary

The challenge

- For which graph properties P does $\chi_P(G; X)$ have the Difficult Point Dichotomy?
- Can we show DPD for infinitely many essentially different graph properties P?
- Can we show DPD for all graph properties P?
- Or at least for graph properties recognizable in (non-deterministic) polynomial time ?

I am tempted to conjecture that the answer is positive.

But I have been proven wrong with previous similar conjectures.

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Graph polynomials with known full complexity spectrum

From GHKMN, Advances in Applied Mathematics,

http://dx.doi.org/10.1016/j.aam.2017.04.005

G-polynomial	$E = \text{EASY}(P)$	$\sharp\text{PHARD}(P)$	OTHER	Reference
$\chi_{trivial}(G;X)$	$E_{trivial} = \mathbf{F}$, u	$\emptyset$	$\emptyset$	trivial
$p_A(G;X)$	$E_{char} = \mathbf{F}$, u	$\emptyset$	$\emptyset$	folklore
$gm(G;X)$	$E_{match} = \{0\}$	$\mathbf{F} - E_{match}$	$\emptyset$	folklore
$\chi(G;X)$	$E_{chrom} = \{0,1,2\}$	$\mathbf{F} - E_{chrom}$	$\emptyset$	Theorem 1.3
$\chi_{harm}(G;X)$	$E_{harm} = \mathbb{N}$, nu	$\mathbf{F} - E_{harm}$	$\emptyset$	Theorem 3.2
$\chi_{convex}(G;X)$	$E_{convex} = \{0,1\}$	$\mathbf{F} - E_{convex}$	$\emptyset$	Theorem 3.6
$\chi_{DU(K_\alpha)}(G;X)$ $\alpha \geq 2$	$E_{DU(K_\alpha)} = \{0,1\}$	$\mathbf{F} - E_{DU(K_\alpha)}$	$\emptyset$	Theorem 3.16

Full complexity spectra, u=uniformly, nu=non-uniformly

Graph polynomials with known discrete complexity spectrum

From GHKMN, Advances in Applied Mathematics,

http://dx.doi.org/10.1016/j.aam.2017.04.005

G-polynomial	$E = \text{EASY}(P)$	$\sharp\text{PHARD}(P)$	OTHER	Reference
$\chi_{edge}(G;X)$	$E_{edge} = \{0,1\}$	$\mathbb{N} - E_{edge}$	$\emptyset$	Theorem 5.2
$\chi_{mcc_t}(G;X)$ $t \geq 2, k \geq 2$	$E_{mcc_t} = \{0,1\}$	$\mathbb{N} - E_{mcc_t}$	$\emptyset$	Theorem 5.8
$\chi_{H-free}(G;X)$	$E_{H-free} = \{0,1\}$	$\mathbb{N} - \{0,1,2\}$	$\{2\}$ $(+)$	Theorem 5.10

Discrete complexity spectra only, H of size 2, $(+)$ only $\mathbf{NP}$-hard is known

Thank you for your attention!

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