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Mathematik

On the Comput. Complexity of the Dirichlet Problem for Poisson's Equation

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Recap: Complexity of Operators

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f:[0;1]→[0;1] polytime computable (⇒ polyn. modulus of continuity)

- $f \to \int_0^1 f(t)\,dt$
  computable in exponential time;
  $\#\mathcal{P}_1$-"complete" [Friedman&Ko'82]
  (cmp. e.g. Valiant'79, Goldsmith&Ogihara&Rothe'98)
- $\int: f \to \int f: (x \to \int_0^x f(t)\,dt)$
  computable in exponential time;
  polytime-computable iff $\mathcal{FP}=\#\mathcal{P}$

even when restricting to $f \in C^\infty$
but for analytic f polytime

- dsolve: $C[0;1]\times[-1;1] \ni f \to z$: $\dot z(t)=f(t,z)$, $z(0)=0$.
  - in general no computable solution $z(t)$
  - for $f\in C^1$ $\mathcal{PSPACE}$-"complete"
  - for $f\in C^k$ $\mathcal{CH}$-"hard"

[Kawamura'10, Kawamura et al]

Poisson's Equation

- [Pour-El&Richards'81, ..., Weihrauch&Zhong'02] In-/computability of the *Wave Equation* (hyperbolic)
- Computability of some *non*linear PDEs: [Yoshikawa'00;Weihrauch&Zhong]

PDE on connected and open set Ω:

$$\Delta u = f \text{ on } \Omega, \quad u = g \text{ on } \partial\Omega$$

- electrostatic / gravitational potential of the charge/ mass distribution f with boundary condition g
- 2nd order, linear, elliptic: homog. $(f,0)$ and inh. $(0,g)$
- 'fundamental' solutions $\ln|\underline{x}|$ (2D) and $1/|\underline{x}|$ (3D)
- 'explicit' Green's functions for various domains,
- solution formula on the complex unit disc; e.g. $g=0$

$$u(z) = -\frac{1}{2\pi} \cdot \int_{|w|\leq 1} \ln \frac{|w-z|}{|w\cdot z^* - 1|} \cdot f(w)\, dA(w)$$

Green's Function in 2D

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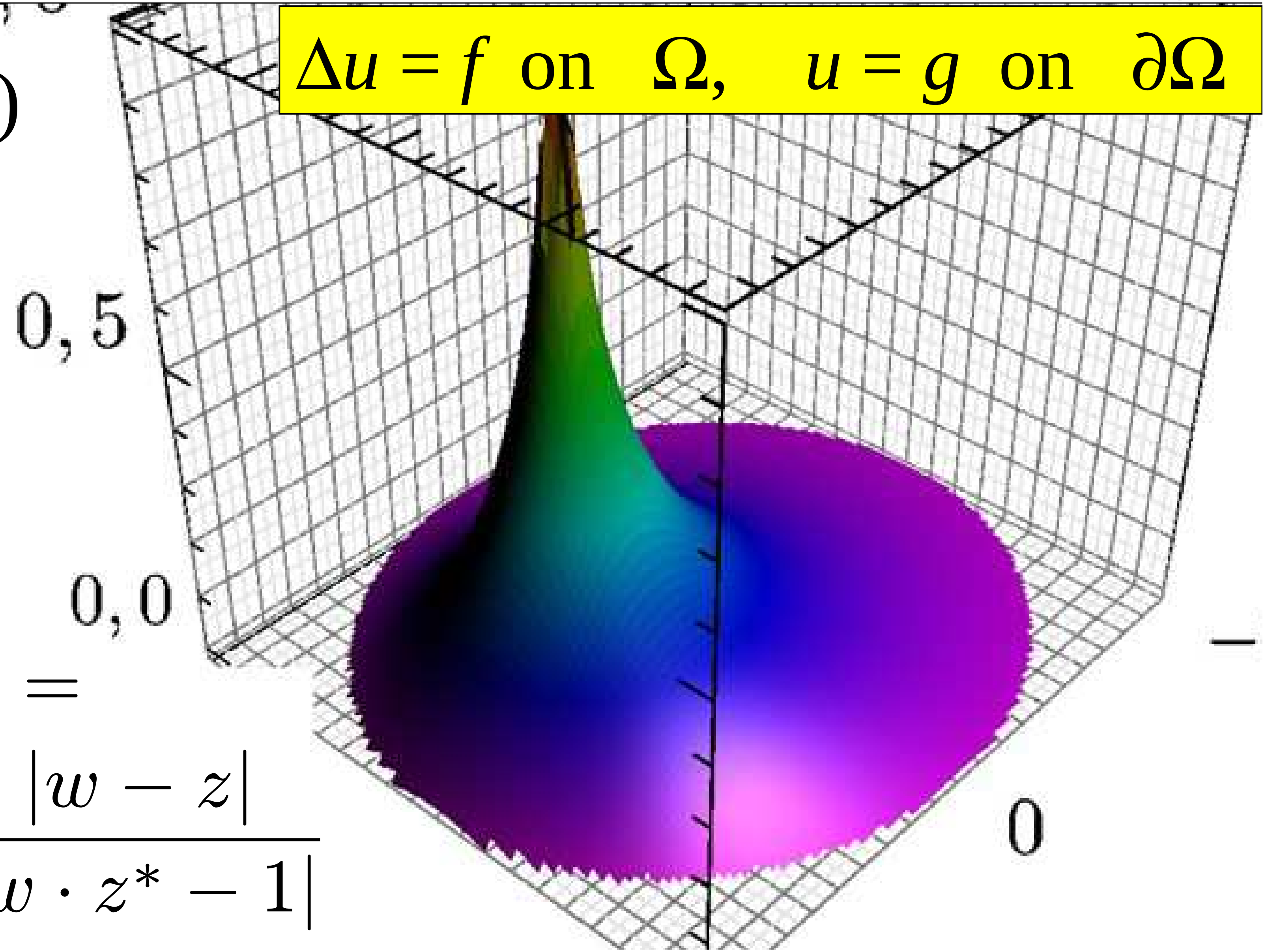
$$\Delta u = f \text{ on } \Omega, \quad u = g \text{ on } \partial\Omega$$

$G(1,w)$



$$G(z,w) = \ln \frac{|w - z|}{|w \cdot z^* - 1|}$$

Results

$\Delta u = f \text{ on } \Omega, \quad u = g \text{ on } \partial\Omega$

Theorem: B_d := closed d-dim. Euklidean unit ball

a) For every polytime $f{:}B_d \to \mathbb{R}$ and $g{:}\partial B_d \to \mathbb{R}$, there exists a unique C^2 solution $u =: \Phi(B_d, f, g)$ and u is computable in exponential time.

b) If $\mathcal{FP} = \#\mathcal{P}$, then u is even polytime computable.

c) There exists a polytime $f \in C^\infty$ such that $u = \Phi(B_d, f, 0)$ is polytime iff $\mathcal{FP} = \#\mathcal{P}$.

d) For $d > 1$ there is a polytime $g \in C^\infty$ s.t. $\Phi(B_d, 0, g)$ is polytime iff $\mathcal{FP} = \#\mathcal{P}_1$

$$u(z) = -\frac{1}{2\pi} \cdot \int_{|w| \le 1} \ln \frac{|w - z|}{|w \cdot z^* - 1|} \cdot f(w)\, dA(w)$$

Proof (Sketch)

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- f,g polyn.modulus of continuity $\Rightarrow$ $u=\Phi(f,g)$ is C^2.
- Bound improper integral (singularity) for $w\to z$.
- Parameter integration over $|w\text{-}z|>2^{-n}$ uniformly feasible in $\#\mathcal{P}$.
- Take polytime $h\in C^\infty[0;1]$ s.t. $\int h$ is "$\#\mathcal{P}$-hard" and let $f(\underline{x}) := h'(|\underline{x}|) \,/\, |\underline{x}|\cdot\ln|\underline{x}|$ radially symmetric.

b) If $\mathcal{FP}=\#\mathcal{P}$, then u is even polytime computable.

there exists a unique C^2 solution $u=:\Phi(f,g)$ and u is computable in exponential time.

$$u(z) = -\frac{1}{2\pi}\cdot\int_{|w|\leq 1} \boxed{\ln\frac{|w-z|}{|w\cdot z^*-1|}}\cdot f(w)\,dA(w)$$

Under further investigation...

Theorem: b+c) $f \to \Phi(B_d, f, 0)$ maps (smooth) polytime to polytime functions iff $\mathcal{FP} = \#\mathcal{P}$.

d) For $d > 1$ there is a polytime $g \in C^\infty$ s.t.

$\Phi(B_d, 0, g)$ is polytime iff $\mathcal{FP} = \#\mathcal{P}_1$

- Uniform: 2nd order polytime W-Reduction
- Close gap w.r.t. g between $\#\mathcal{P}_1$ and $\#\mathcal{P}$
- Consider domains other than balls B_d
- 2D $\to$ complexity of the Riemann mapping
- other (nonlinear) PDEs $\to$ Navier-Stokes!