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Knot theory: Algorithms, complexity and computation

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National Institute of Informatics
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Knot theory:
Algorithms, complexity and computation

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Knot theory is, at its most basic level, concerned with the topology of closed loops in 3-dimensional space. This mathematical subject is inherently algorithmic: old and fundamental questions include how to test whether two knots are topologically equivalent (isotopic), or how to enumerate all distinct knots up to a given complexity.

Great progress has been made on these problems mathematically, from Haken's solution to the unknottedness problem half a century ago [7], to the Gordon-Luecke theorem which converts the broader knot equivalence problem into an algorithmic problem on triangulated 3-manifolds [6].

Nevertheless, algorithmic problems on knots remain challenging for those who wish to do real computations. For instance, the best-known algorithms for just the "simple" problem of testing unknottedness are exponential-time, and a full enumeration of prime knots has only been carried out for knots of ≤ 16 crossings [9]. Such problems have now attracted the attention of researchers in algorithms and complexity, as well as other branches of computer science. Examples of current questions include:

- *How do algorithmic questions from knot theory fit into the hierarchy of complexity classes?*
  Testing unknottedness is known to be NP [8], and also co-NP assuming the generalised Riemann hypothesis [10]. However, is it in P? Is it NP-complete? Is it fixed-parameter tractable for some natural parameter?

- *What are the best methods and heuristics for solving knot problems in practical software?*
  There are many invariants, geometric methods and heuristic techniques that are extremely effective in practice but do not guarantee a conclusive solution [1, 5, 11]—can we prove that these work for average or generic inputs? For a conclusive solution, methods from operations research have proven remarkably effective in testing unknottedness [3], but can these be generalised?

- *How can we efficiently generate and effectively manage databases of knots and their properties?*
  The excellent online sources such as *KnotInfo* [4] and the *Knot Atlas* [2]

are still relatively small (e.g., the exhaustive *KnotInfo* database contains only the first 2977 prime knots), and generating new data will require a careful interplay between large-scale combinatorial enumeration and complex topological decision problems. As databases grow in scale we must also address issues of computing complex properties in bulk, and effectively exploring and mining the resulting data.

The purpose of this meeting was to bring together experts in computational and algorithmic knot theory. Speakers ranged across the spectrum from pure mathematicians to theoretical computer scientists and practical software developers. In particular, we drew on expertise from a range of inter-related areas:

- development of mathematical software;
- generation and management of data collections;
- computational complexity of knot problems;
- numerical algorithms for studying the geometries of knots;
- discrete algorithms in computational geometry and integer programming;
- visualisation of knots;
- geometric topology, in particular 3-manifold and 4-manifold topology.

By combining these areas of expertise, the aim of the meeting was to generate new ideas and spawn new long-term projects that could create a clear theoretical picture of the intrinsic algorithmic difficulty of problems in knot theory, and set a new standard for what software can achieve in this rich and complex problem domain.

Overview of Talks

Genus bounds for normal surfaces

Stephan Tillmann, The University of Sydney, Australia

This is joint work with William Jaco, Jesse Johnson and Jonathan Spreer. We give a sharp bound on the genus of a normal surface in a triangulated compact, orientable 3-manifold in terms of the quadrilaterals in its cell decomposition. In addition, we describe two applications of this bound.

Meridional and non-meridional epimorphisms between knot groups

Masaaki Suzuki, Meiji University, Japan

We consider epimorphisms between knot groups. A homomorphism between knot groups is called meridional if it preserves meridians. The existence of a meridional epimorphism induces a partial order on the set of prime knots. In this talk, we determine this partial order on the set of prime knots with up to 11 crossings and sketch how to prove it by several examples. The key tool is the twisted Alexander polynomial, and we need to make use of the computer in order to calculate this invariant. Furthermore, we show infinitely many examples of non-meridional epimorphisms.

On pseudomodular groups

Yasushi Yamashita, Nara Women's University, Japan

Let G be a Fuchsian group. If the set of parabolic fixed points of G is equal to the union of the set of rational numbers and the infinity $1/0$, and G is not commensurable to the modular group $PSL(2, \mathbb{Z})$, then G is called a pseudomodular group. Long and Reid studied pseudomodular groups and gave some examples. In this talk, we review their paper and show some possibly new examples of them.

Topological algorithms for graphs on surfaces

Éric Colin de Verdière, CNRS, École normale supérieure, Paris, France

Given a compact surface with a suitable notion of metric, how can we transform it into a topological disk by cutting it as little as possible? How to compute a shortest non-contractible closed curve on a surface? How to tighten as much as possible a closed curve on a surface up to homotopy? The aim of this talk is to survey algorithms for these problems from computational topology.

Includes works by Jeff Erickson, Sariel Har-Peled, Arnaud de Mesmay, Kim Whittlesey, and the speaker.

A census of knots in homotopy 4-spheres

Ryan Budney, University of Victoria, Canada

I will describe an ongoing project to form a census of all triangulated 4-manifolds triangulable with 6 or fewer 4-dimensional simplices. The table of knots in homotopy 4-spheres is the closest to completion, which this talk will focus on. A new 2-knot type was discovered that helps resolve the types of fundamental groups that appear as 1, 2 and 3-knot exteriors.

3-manifolds algorithmically bound 4-manifolds

Sam Churchill, University of Victoria, Canada

In their 2008 paper Dylan Thurston and Francesco Costantino demonstrate that, given a 3-manifold M of complexity n, one may construct a 4-manifold bounded by M of complexity $O(n^2)$. Here, the complexity of a piecewise-linear manifold is the minimum number of highest dimensional simplices in a triangulation of that manifold.

The proof of this fact is constructive and begins with a projection M to $\mathbb{R}^2$. The preimage of a generic point is the disjoint union of circles which bounds a collection of disks. Extending this construction across codimension 1 and 2 singularities produces the desired 4-manifold.

The object central to this construction is called a shadow and is completely combinatorial information. The combinatorics of a triangulation of M completely determines its shadow. The goal of this research is to extend the construction by Thurston and Costantino into an algorithm.

Normal 3-manifolds in triangulated 4-manifolds

Joachim Hyam Rubinstein, The University of Melbourne, Australia

This is joint work with Ben Burton and Bell Foozwell. Normal surfaces have proved to be a pivotal tool in algorithms for 3-manifolds. Analogously, one can define normal 3-manifolds in triangulated 4-manifolds. Given an aspherical π_1-injective 3-manifold M embedded in a triangulated 4-manifold W, by a suitable normalisation process, a normal 3-manifold M' is obtained for which the image of $\pi_1(M')$ is the same as the image of $\pi_1(M)$. There is a degree one mapping from M' to M. The space of normal 3-manifolds has a canonical basis consisting of edge solutions and double pentachoron solutions, similar to the situation in dimension 3. One can use prism coordinates to reduce the dimension of the solution space similar to the use of quadrilateral coordinates in dimension three. Finally, there is an algorithm to decide if a triangulated 4-manifold W can be written as $W' \cup W''$, where W' and W'' intersect in their common boundary M and the image of $\pi_1(M)$ is trivial, assuming that $\pi_1(W)$ is a non-trivial free product, so long as $\pi_1(W)$ has solvable word problem.

Software session

Benjamin Burton, The University of Queensland, Australia
Neil Hoffman, The University of Melbourne, Australia
Takuya Sakasai, The University of Tokyo, Japan
Jonathan Spreer, The University of Queensland, Australia

Various speakers give a discussion and live demonstration of topological software packages, including *HIKMOT*, *Regina*, *simpcomp*, *SnapPy*, and *Teruaki.*

The moduli space of pentagons

Sadayoshi Kojima, Tokyo Institute of Technology, Japan

We show that the moduli space of marked pentagons up to similarity is naturally identified with a hyperbolic surface of genus 4 made of 24 right angled regular pentagons, and then we discuss its algebraic representation and generalizations for other polygons.

A talk in 3 acts: Verified computations, L-space surgeries, and exceptional fillings

Neil Hoffman, The University of Melbourne, Australia
Kimihiko Motegi, Nihon University, Japan

By most accounting methods, hyperbolic manifolds are the most common and yet least understood manifolds. However, it is often difficult to prove that a manifold given is hyperbolic.

The first part of this talk will describe a new method for verifying that a hyperbolic structure exists for a given triangulated manifold.

The second part of the talk will describe an interesting knot complement that seems to have a hyperbolic structure and admits no exceptional surgeries. However, this knot complement is interesting from the perspective of Heegaard-Floer homology, namely that of L-space surgeries.

The final part of the talk uses computer software based on the first part that finds candidates for non-hyperbolic fillings and verifies that indeed there are no exceptional fillings on the knot complement from part 2.

Act 1: Presentation of work by Hoffman, Ichihara, Kashiwagi, Masai, Oishi, Takayasu "Verified computations for hyperbolic 3-manifolds", describing how to rigorously determine small neighborhoods of the tetrahedral shapes of a triangulated hyperbolic 3-manifold.

Act 2: Presentation of work by Motegi and Toki providing an example of a knot that admits L-space surgeries, but seems to have no exceptional surgeries.

Act 3: Presentation of `fef.py`, an extension of "Verified computations for hyperbolic 3-manifolds", Python code written concurrently by Ichihara and Masai ("Exceptional surgeries on alternating knots"), and Martelli, Petronio and Roukema ("Exceptional Dehn surgery on the minimally twisted five-chain link)".

Tightness for triangulations

Jonathan Spreer, The University of Queensland, Australia

In differential geometry tightness generalises the notion of convexity to manifolds distinct from the sphere or the ball. In combinatorial topology the discrete version of this concept leads to a very special class of tight triangulations with particularly nice properties which is still far from being understood. We will present recent developments in the field focusing on dimension three and discuss several interesting challenges in the field from the viewpoint of both topology and complexity theory.

Exotic components in linear slices of quasi-Fuchsian groups

Yuichi Kabaya, Kyoto University, Japan

The linear slice of quasi-Fuchsian punctured torus groups is defined by fixing the length of some simple closed curve to be a fixed positive real number. It is known that the linear slice is a union of disks, and it has one 'standard' component containing Fuchsian groups. Komori-Yamashita proved that there exists a non-standard component if the length is sufficiently large. In this talk, we give another proof based on the theory of complex projective structures, and show some pictures related to our proof.

The link volume of 3-manifolds and cosmetic surgery on links

Yo'av Rieck, University of Arkansas, USA

Open problem session

The following questions were raised during the problem session on 30 April 2015:

- Is there a polynomial-time algorithm for unknot recognition? As a starting point, is unknot recognition fixed-parameter tractable in a useful parameter? Treewidth could be a good candidate. *(Burton, Spreer)*
- Can we implement the solution to the word problem for 3-manifold groups? Can we solve 3-manifold problems by representing fundamental groups in $PSL(2, \mathbb{C})$? Good candidate problems: testing whether manifolds fibre over S^1. *(Rieck)*
- Can we find 2-knots with hyperbolic and/or aspherical exteriors? In particular, embeddings of the 2-torus into S^4? Ratcliffe has examples, but it would be nice to construct more. *(Rubinstein)*
- Every 3-manifold admits a hyperbolic knot. Is it possible that this is true for 4-manifolds? *(Rieck)*
- Find examples of embeddings of genus ≥ 2 surfaces into S^4 with $K(\pi, 1)$ exterior. *(Kawauchi)*
- Set up a theory of normal 2-manifolds in triangulated 4-manifolds. Normalisation is easy, but the algorithmic framework becomes hard—it becomes more algebraic geometry than integer programming. *(Budney)*
- Build a practical algorithmic framework for working with immersed normal surfaces in 3-manifolds. There are underlying hard problems to deal with and no structure theorems, but this does not need to kill off the approach entirely. *(Burton)*
  Such a framework could be used with several underlying problems, such as testing for incompressibility, testing for embedded covers, projections of embeddings in covering spaces, ... *(Rubinstein, Tillmann)*
- What is the smallest volume of an orientable hyperbolic 3-manifold with positive Betti number? The smallest known is the figure eight knot complement. *(Kojima)*
  What is the smallest volume non-orientable closed hyperbolic manifold? Experimentally, this appears to be the 10-tetrahedron hyperbolic manifold with the same volume as the figure eight knot complement. *(Hoffman)*

Schedule

28th April (Monday)

- 9:15–9:30: Welcome and information
- 9:30–10:30: Stephan Tillmann: *Genus bounds for normal surfaces*
- 11:00–12:00: Masaaki Suzuki: *Meridional and non-meridional epimorphisms between knot groups*
- 14:00–15:00: Yasushi Yamashita: *On pseudomodular groups*
- 16:00–17:00: Éric Colin de Verdière: *Topological algorithms for graphs on surfaces*

29th April (Tuesday)

- 9:00–9:45: Ryan Budney: *A census of knots in homotopy 4-spheres*
- 9:45–10:30: Sam Churchill: *3-manifolds algorithmically bound 4-manifolds*
- 11:00–12:00: Joachim Hyam Rubinstein: *Normal 3-manifolds in triangulated 4-manifolds*
- 14:00–15:35: Software session:
  - 14:00–14:20: simpcomp: *Jonathan Spreer*
  - 14:25–14:45: SnapPy & HIKMOT: *Neil Hoffman*
  - 14:50–15:10: Teruaki: *Takuya Sakasai*
  - 15:15–15:35: Regina: *Benjamin Burton*
- 16:30–17:30: Sadayoshi Kojima: *The moduli space of pentagons*

30th April (Wednesday)

- 9:00–10:30: Neil Hoffman & Kimihiko Motegi: *A talk in 3 acts: Verified computations, L-space surgeries, and exceptional fillings*
- 11:15–12:00: Problem session
- 13:30–19:30: Excursion and banquet

1st May (Thursday)

- 9:30–10:15: Jonathan Spreer: *Tightness for triangulations*
- 10:45–11:15: Yuichi Kabaya: *Exotic components in linear slices of quasi-Fuchsian groups*
- 11:15–12:00: Yoav Rieck: *The link volume of 3-manifolds and cosmetic surgery on links*

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