Sketching Graphs

Kook Jin Ahn *University of Pennsylvania*
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- *Rich Theory:* Related to compressed sensing and sparse recovery, dimensionality reduction and metric embeddings, ...

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! No cheating! Assume M is finite precision etc.

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- In semi-streaming, want to process graph defined by edges $e_1, \ldots, e_m$ with $\tilde{O}(n)$ memory and reading sequence in order.

[Muthukrishnan 05; Feigenbaum, Kannan, McGregor, Suri, Zhang 05]

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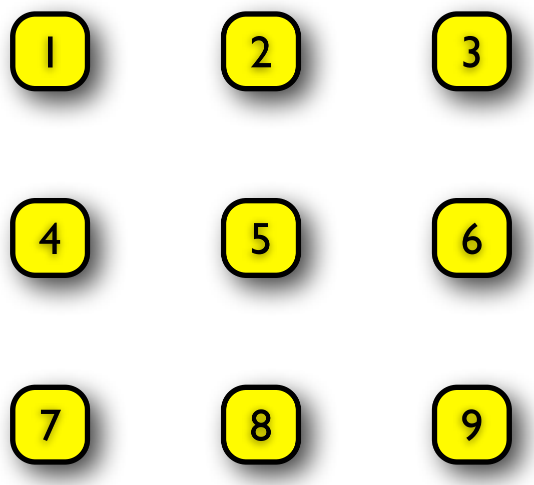

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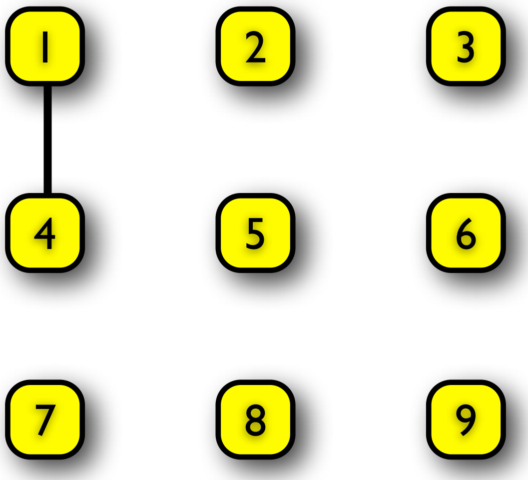

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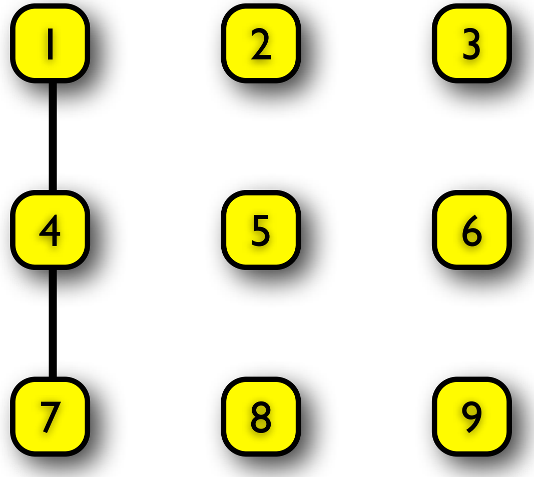

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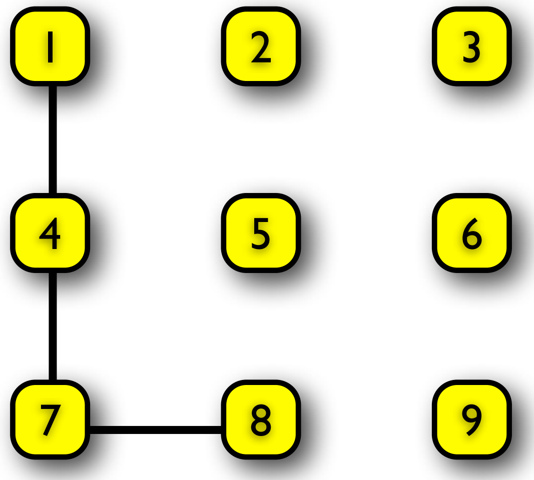

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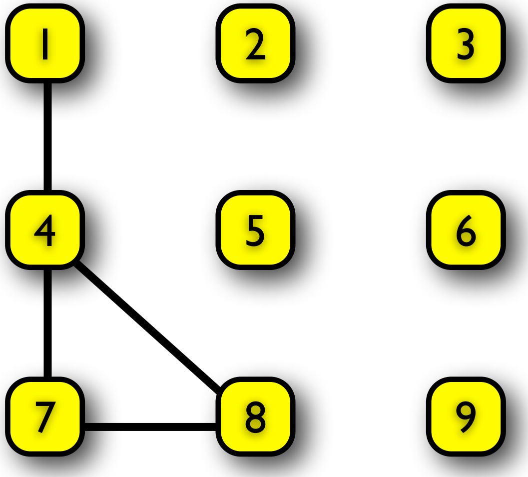

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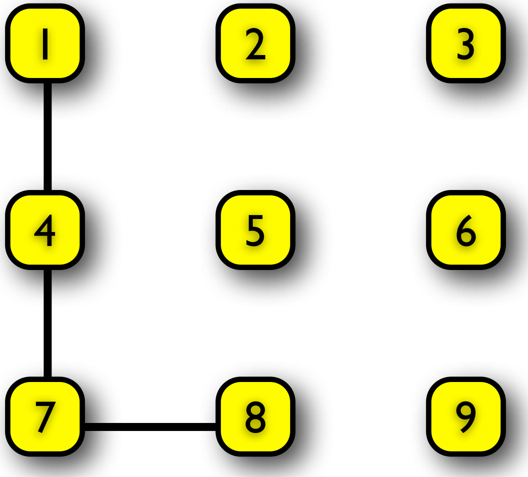

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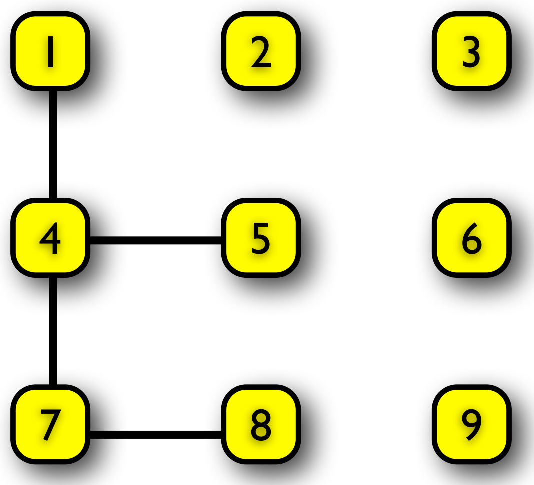

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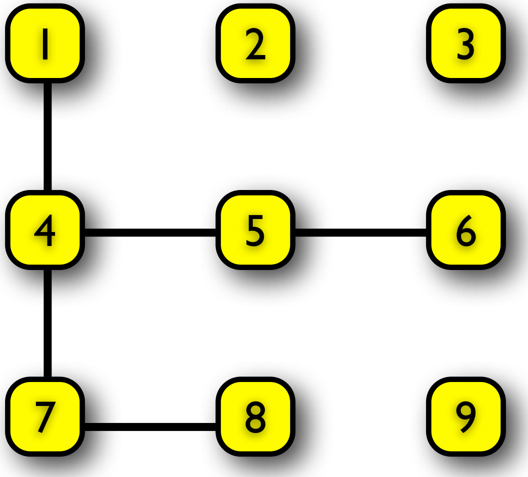

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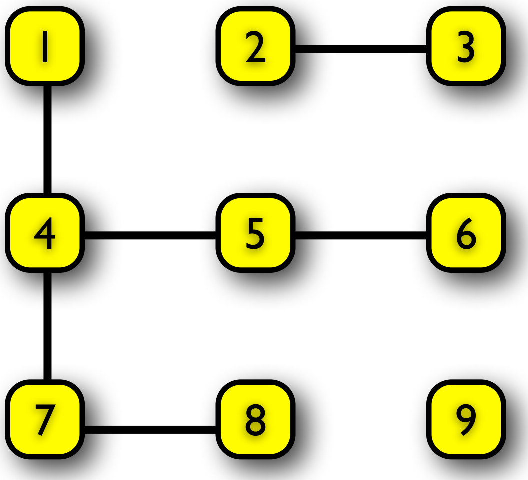

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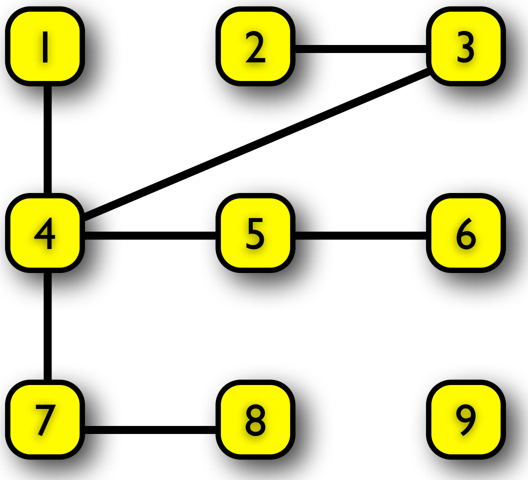

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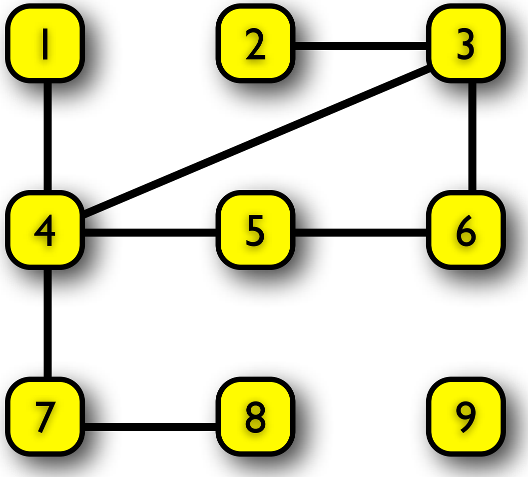

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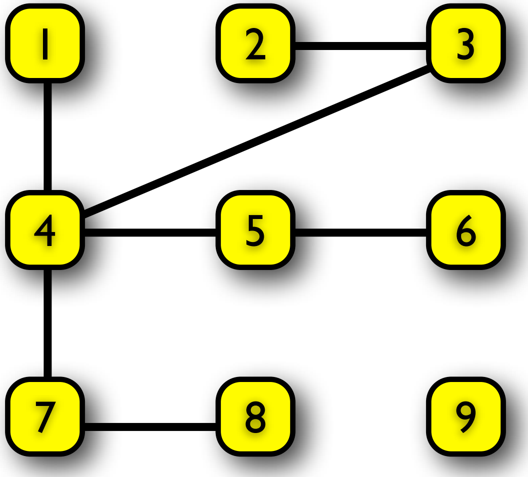

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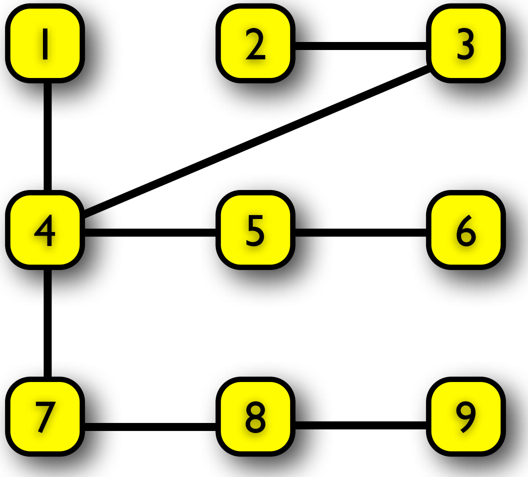

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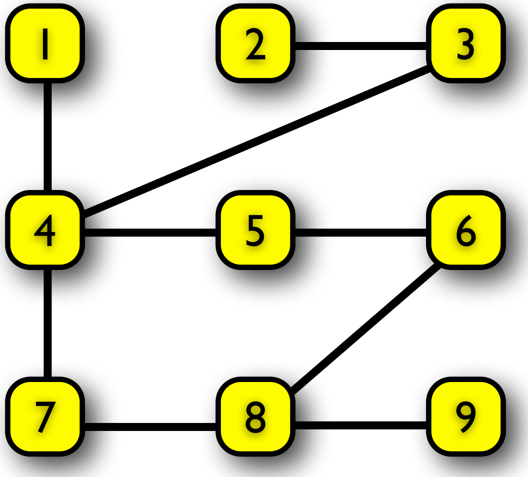

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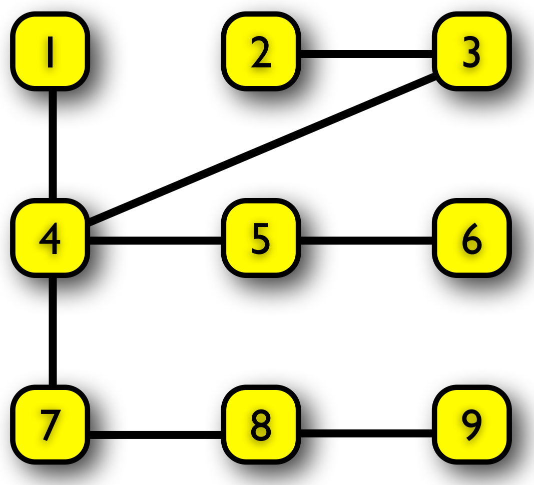

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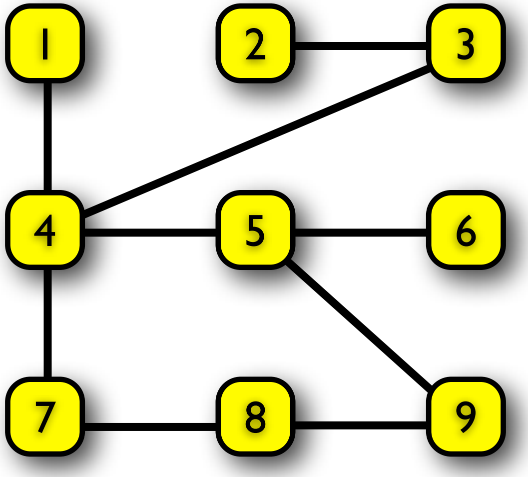

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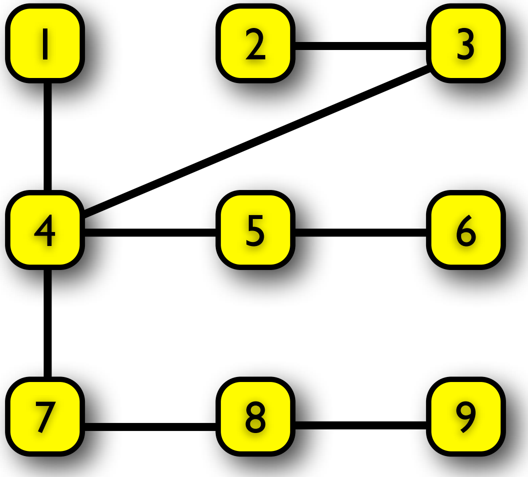

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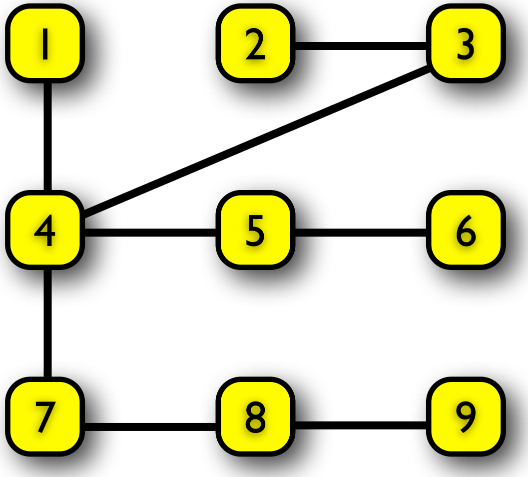

- *Sketches:* To delete e from G: update $MA_G \rightarrow MA_G - MA_e = MA_{G-e}$

Why? Distributed Processing

Input: G=(V,E)

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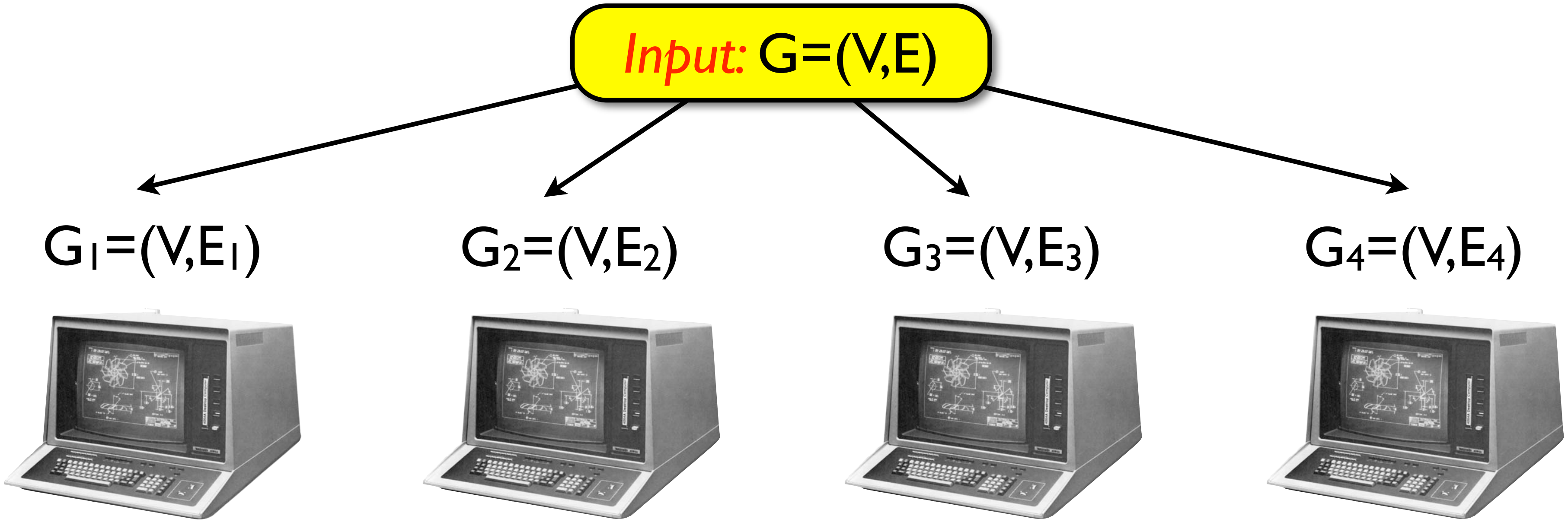

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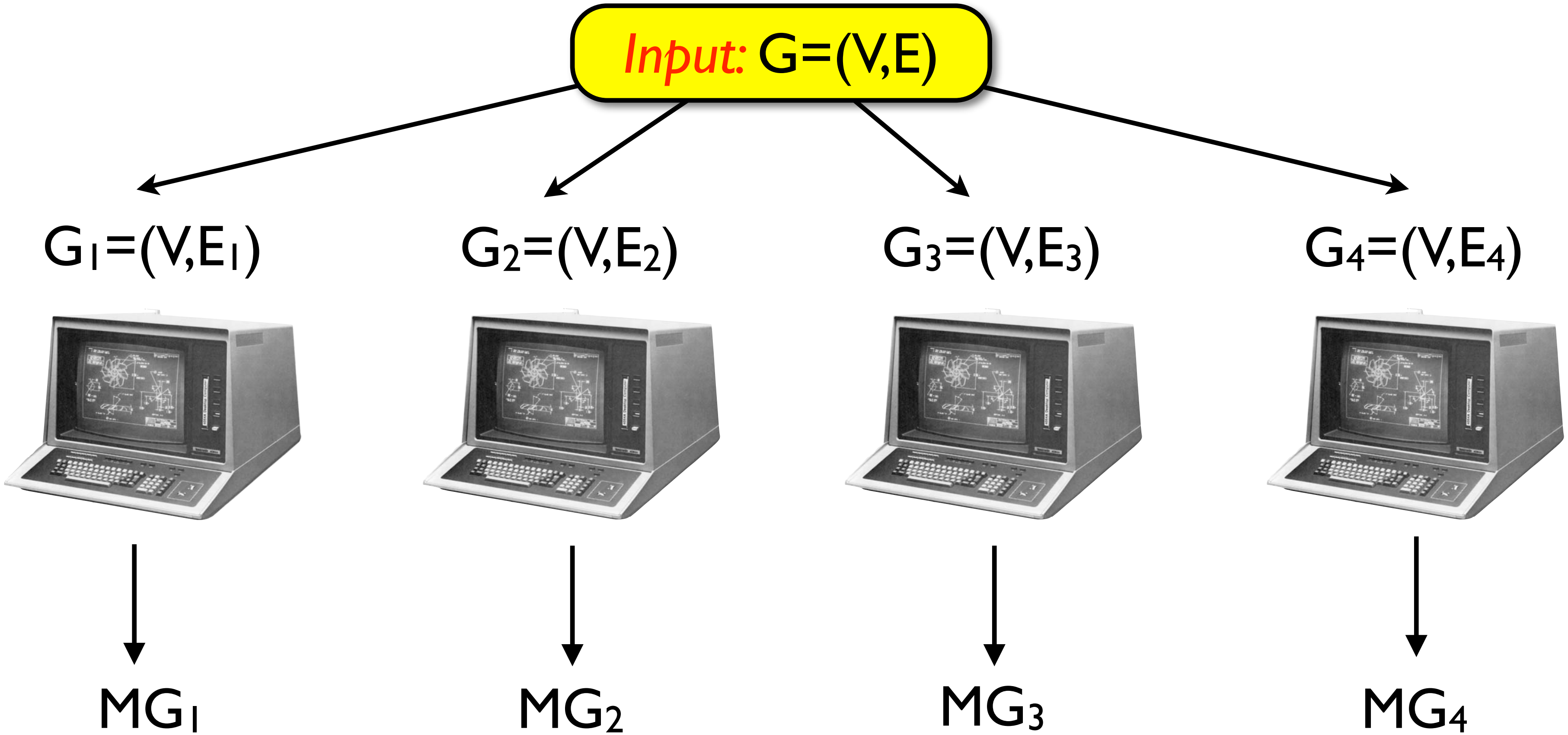

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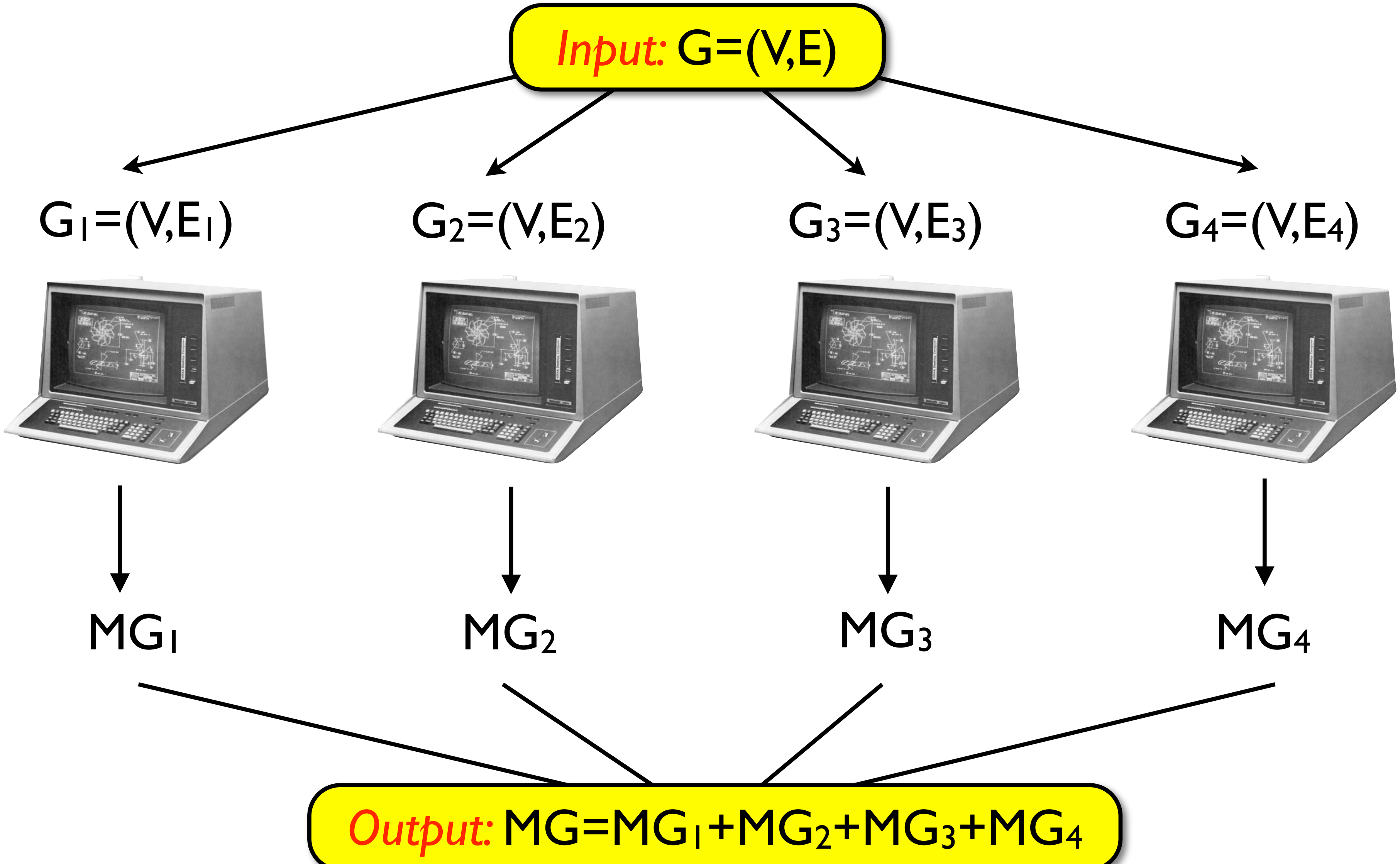

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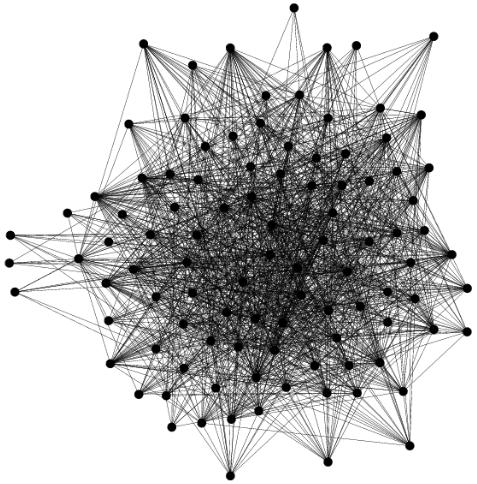

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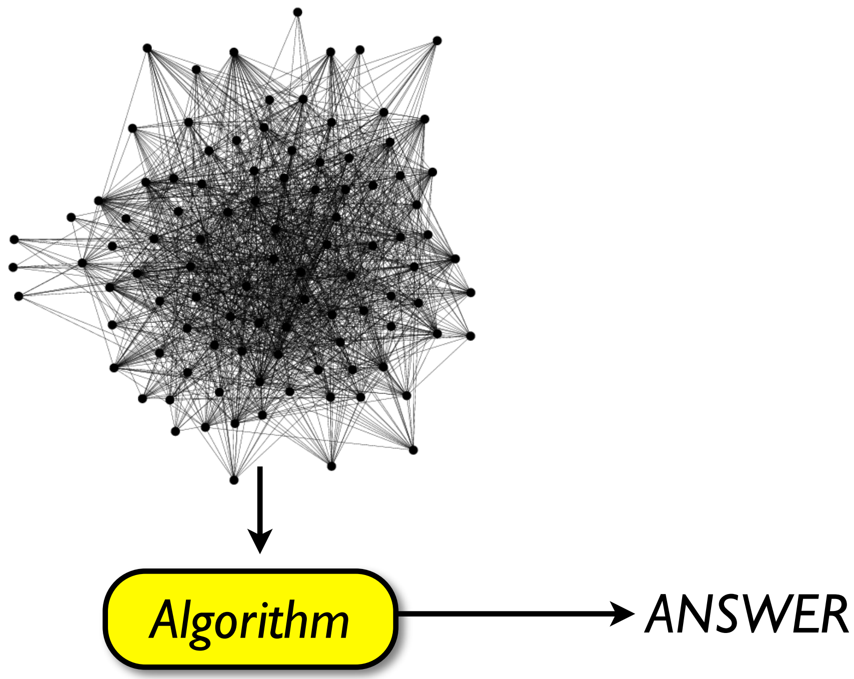

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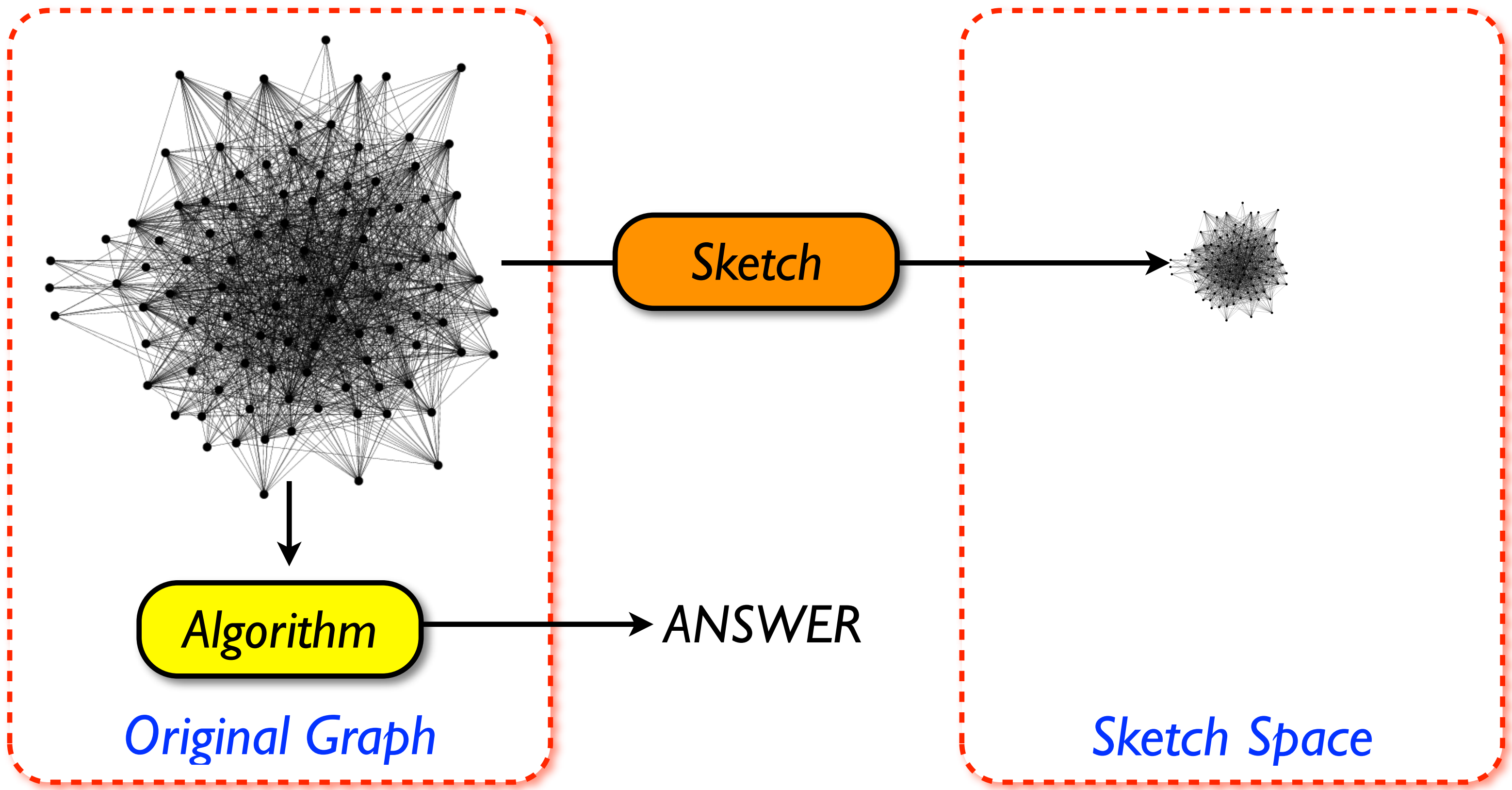

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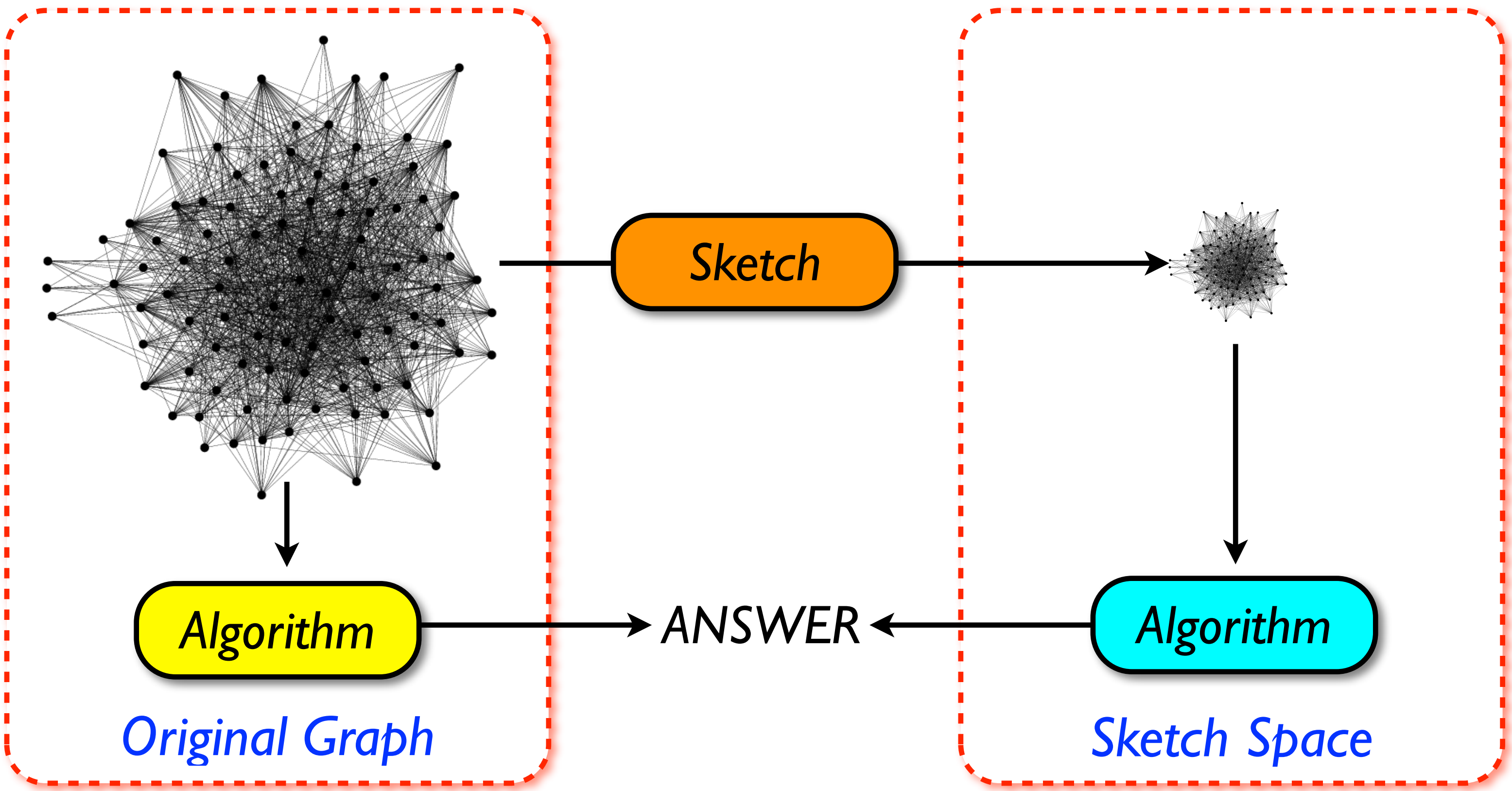

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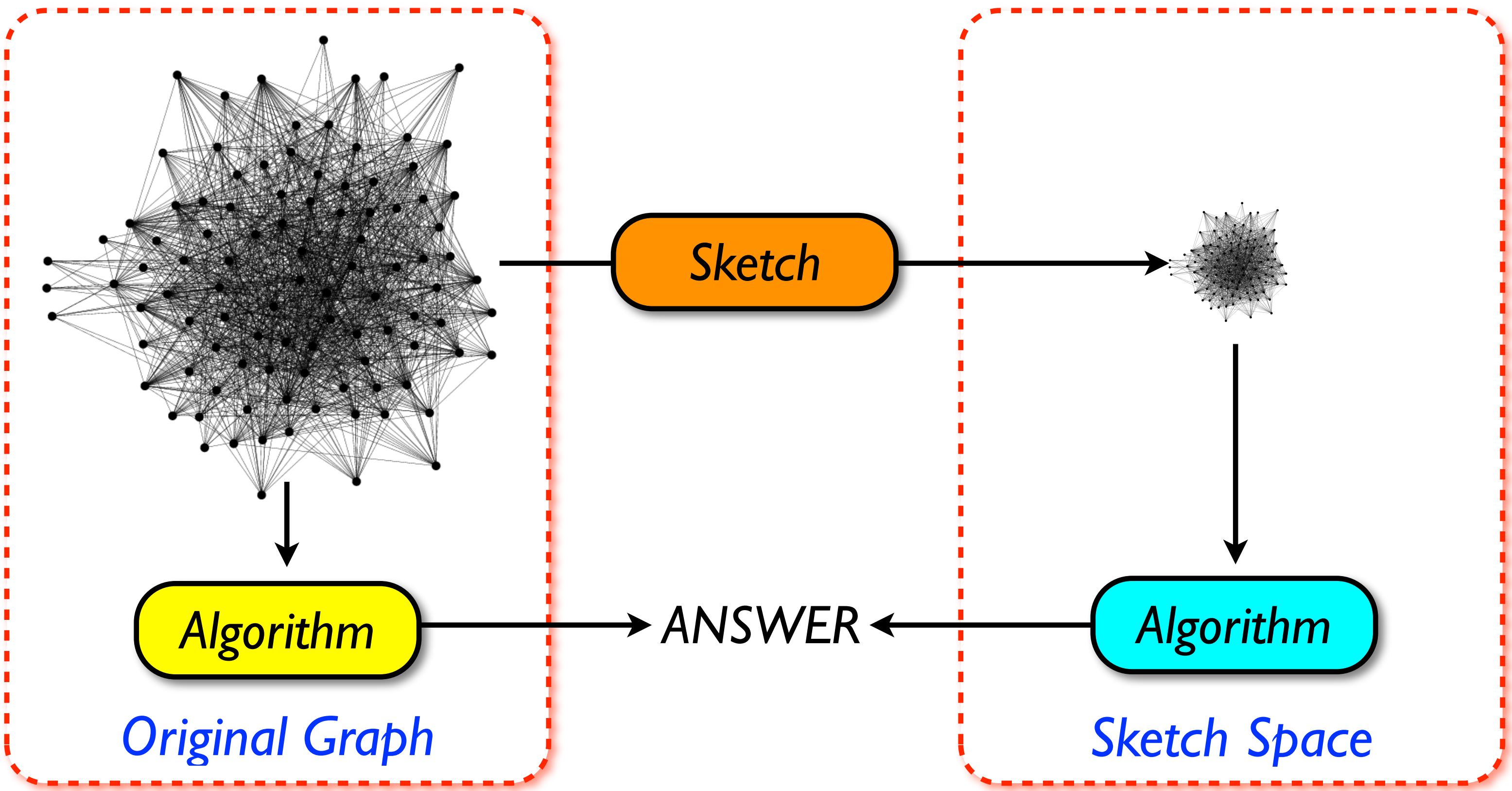

- *Catch:* Sketch must be homomorphic for algorithm operations.

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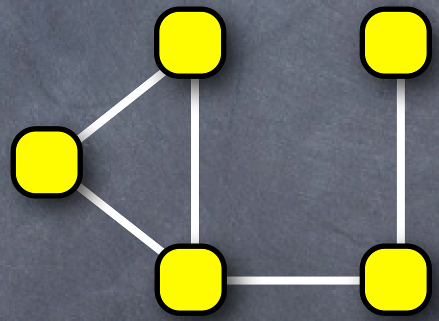

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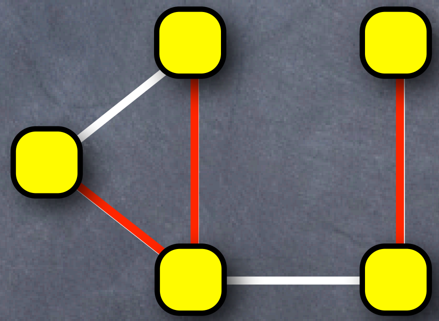

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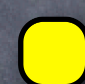

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- Lemma: Takes O(log n) steps and selected edges include spanning forest.

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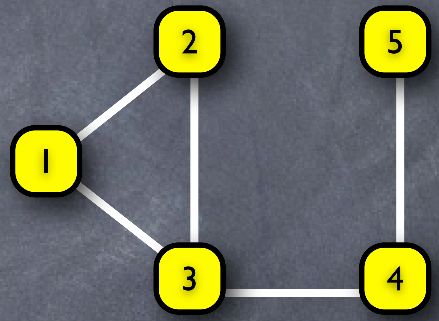

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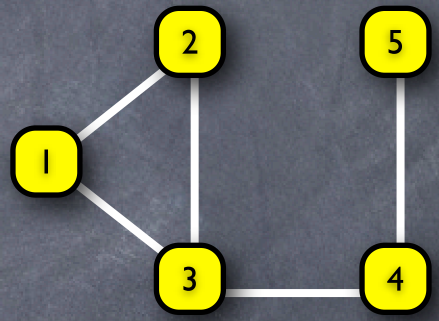
$\mathbf{a}_1 =$	1	1	0	0	0	0	0	0	0	0



Ingredient 2: Graph Representation

- For node i, let a_i be vector indexed by node pairs. Non-zero entries: $a_i[i,j]=1$ if $j>i$ and $a_i[i,j]=-1$ if $j<i$.
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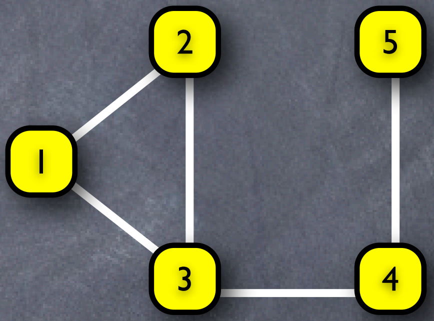

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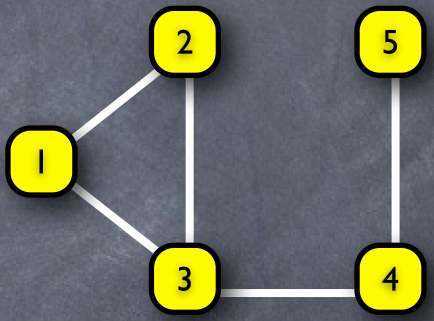

- Lemma: For any subset of nodes $S \subset V$,

$$\text{support}\left(\sum_{i \in S} \mathbf{a}_i\right) = E(S, V \setminus S)$$

Ingredient 3: l_0-Sampling

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- Lemma: Exists random $C \in R^{d \times m}$ with $d=O(\log^2 m)$ such that for any $a \in R^m$

$$C\mathbf{a} \longrightarrow e \in \text{support}(\mathbf{a})$$

with probability 9/10.

[Cormode, Muthukrishnan, Rozenbaum 05; Jowhari, Saglam, Tardos 11]

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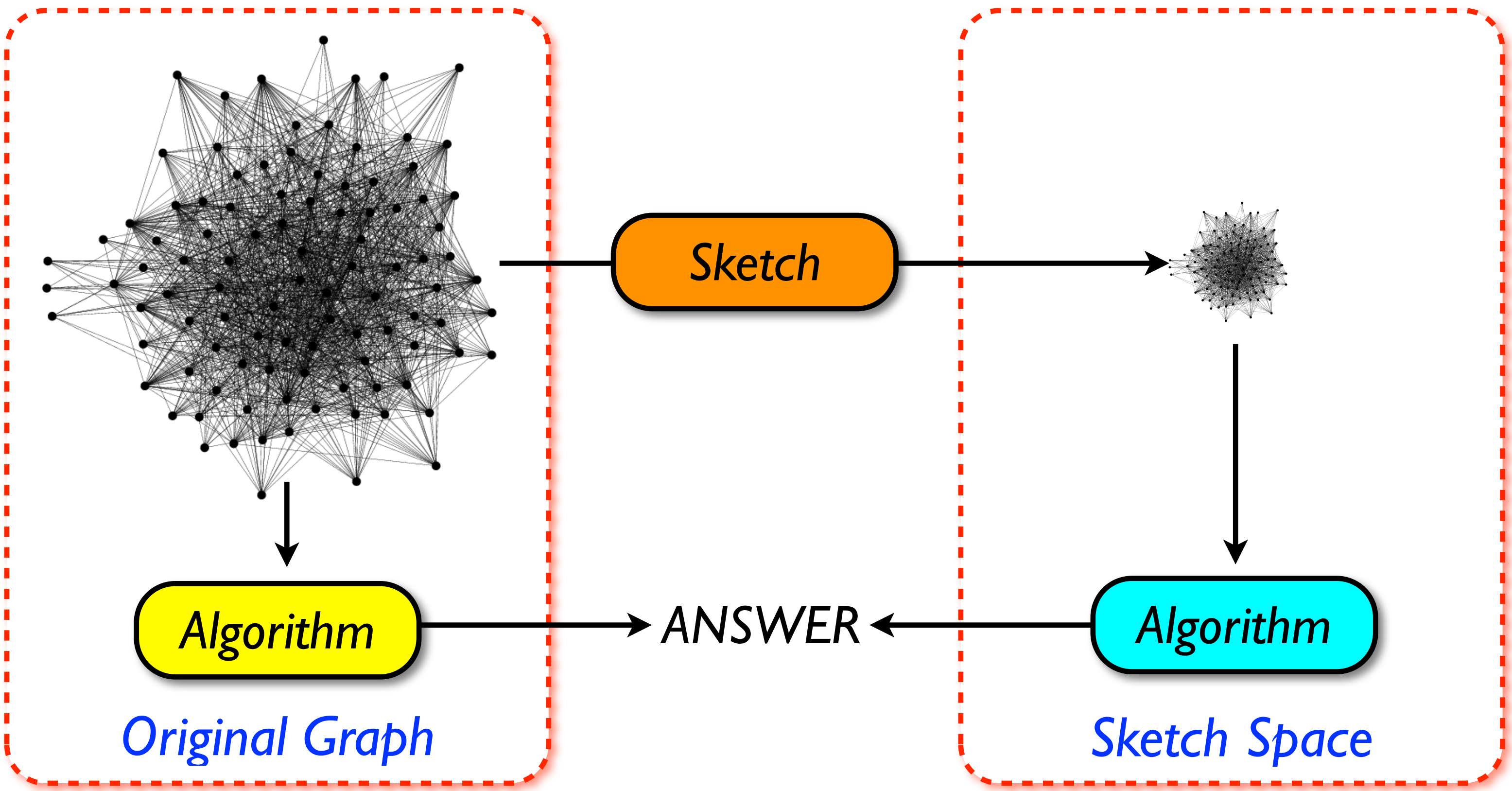
Connectivity

- *Thm:* Can check connectivity with $O(n\log^3 n)$-size sketch.
- *Main Idea:* *a)* Sketch! *b)* Run Algorithm in Sketch-Space



- *Catch:* Sketch must be homomorphic for algorithm operations.

a) Connectivity

b) Applications

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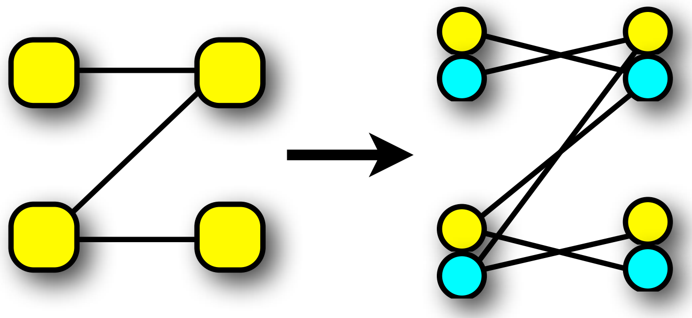

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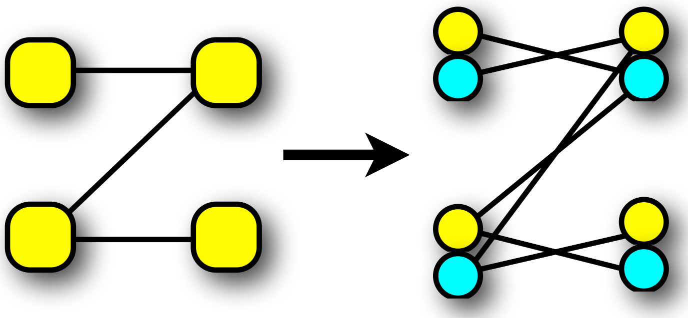

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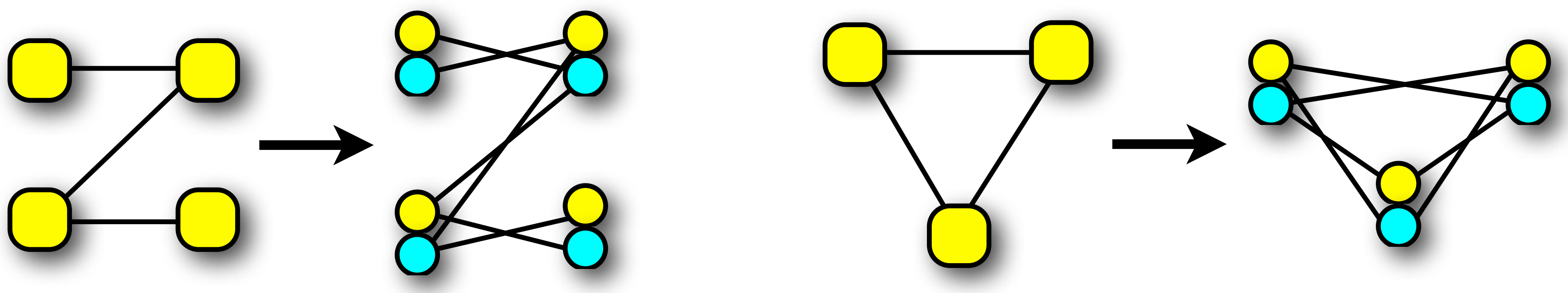

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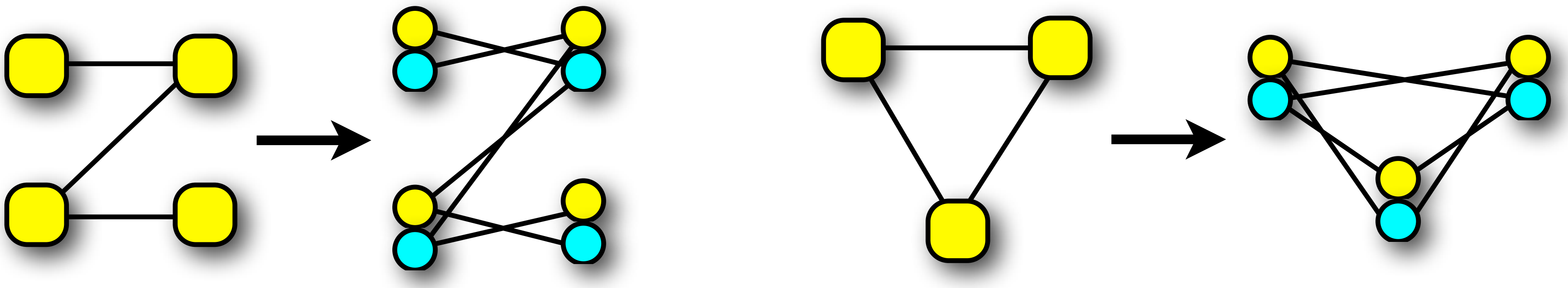

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- *Thm:* Can find exact MST in dynamic semi-streaming model using O(log n/log log n) passes.

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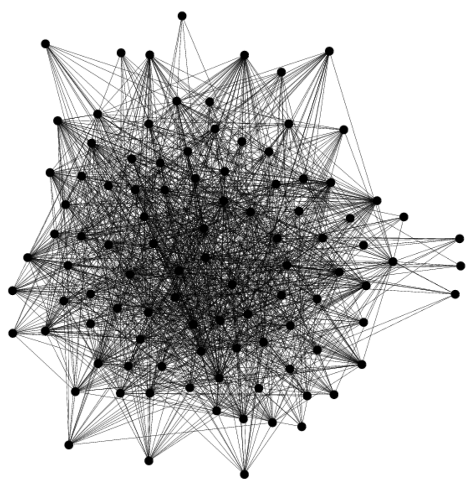
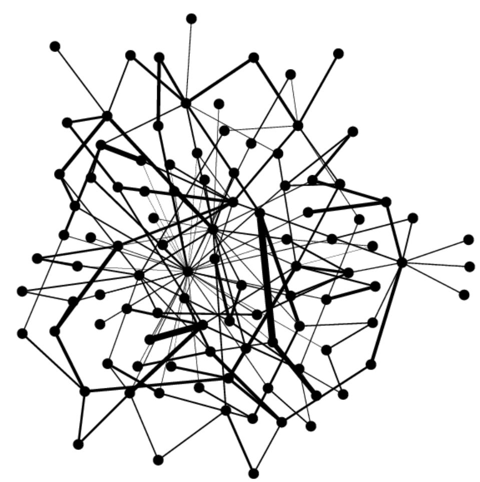


Original Graph



Sparsifier Graph

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- Lemma: $G(V,F_1+\ldots+F_k)$ is k-connected iff $G(V,E)$ is.

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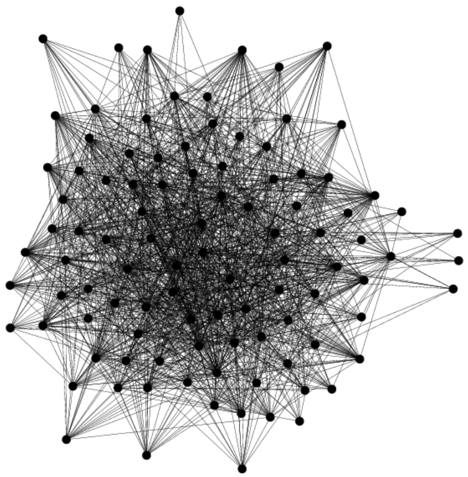

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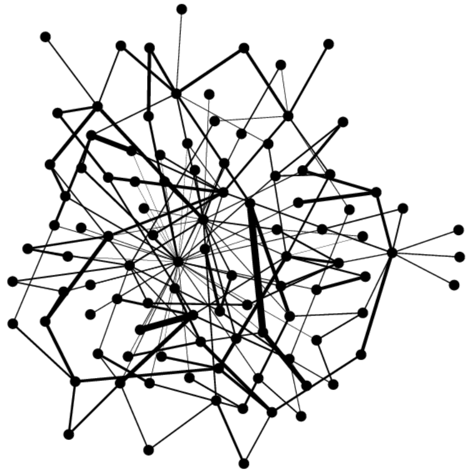

Sparsifier Graph

Summary

- *Graph Sketches:* Initiates the study of linear projections that preserve structural properties of graphs. Application to dynamic-graph streams and are embarrassingly parallelizable.
- *Properties:* Connectivity, sparsifiers, spanners, bipartite, minimum spanning trees, small cliques, matchings, ...

ありがとう！