Combining Proofs and Programs

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Dependently Typed Programming
Shonan Meeting Seminar 007
The TRELLYS project
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A collaborative project to design a statically-typed functional programming language based on dependent type theory.
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A collaborative project to design a statically-typed functional programming language based on dependent type theory.

Work-in-progress
Growing a new language

Trellys Design strategy: Start with general purpose, call-by-value, functional programming language and strengthen its type system.
Why call-by-value?

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Why call-by-value?

- Have to choose something. With nontermination, the order of evaluation makes a difference.
- Good cost model. Programmers can predict the running time and space usage of their programs.
- Distinction between values and computations built into the language. Variables stand for values, not computations.
Programming language vs. logic

Even in the presence of nontermination, a call-by-value dependently-typed programming language provides partial correctness.

**Theorem (Syntactic type soundness)**

\[ \text{If } \vdash^P a : A \text{ then either } a \text{ diverges or } a \sim^* v \text{ and } \vdash^P v : A. \]
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**Theorem (Syntactic type soundness)**

\[
\text{If } \vdash^P a : A \text{ then either } a \text{ diverges or } a \rightsquigarrow^* v \text{ and } \vdash^P v : A.
\]

A dependently-typed logic provides total correctness.

**Theorem (Termination)**

\[
\text{If } \vdash^L a : A \text{ then } a \rightsquigarrow^* v \text{ and } \vdash^L v : A.
\]
Partial correctness

Type soundness alone gives a logical interpretation for \textit{values}.

\[ \vdash^P a : \Sigma x : \text{Nat}. \text{even } x = \text{true} \]

If \( a \) terminates, then it \textit{must} produce a pair of a natural number and a \textit{proof} that the result is even. Canonical forms says the result must be \((i, \text{join})\), where \textit{even} \( i \sim^* \text{true} \) by inversion.
Partial correctness

Type soundness alone gives a logical interpretation for *values*.

$$\vdash^P a : \Sigma x : \text{Nat}. \text{even } x = \text{true}$$

If $a$ terminates, then it *must* produce a pair of a natural number and a *proof* that the result is even. Canonical forms says the result must be $(i, \text{join})$, where $\text{even } i \rightsquigarrow^* \text{true}$ by inversion.

But, implication is bogus.

$$\vdash^P a : \Sigma x : \text{Nat}. (\text{even } x = \text{true}) \rightarrow (x = 3)$$
Partial correctness is not enough.

- Implication is useful
- Can’t compile this language efficiently (have to run “proofs”)
- “Proof” irrelevance is fishy
- Users are willing to work harder for stronger guarantees
A logical language

- But, some programs do terminate. There is a terminating, logically-consistent logic hiding in a dependently-typed programming language.
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New typing judgement form:

\[ \Gamma \vdash^\theta a : A \quad \text{where} \quad \theta ::= L \mid P \]
Subsumption

Many rules are shared.

\[
\frac{\Gamma \vdash x : \theta \quad A \in \Gamma}{\Gamma \vdash x : A} \quad \frac{\Gamma \vdash \theta \quad b : \text{Nat}}{\Gamma \vdash \theta \quad S \ b : \text{Nat}}
\]
Many rules are shared.

$$\Gamma \vdash x : \theta A \in \Gamma \quad \Gamma \vdash b : \text{Nat}$$

$$\Gamma \vdash \theta x : A$$

$$\Gamma \vdash \theta S \ b : \text{Nat}$$

Programmatic language allows features (general recursion, type-in-type, abort etc.) that do not type check in the logical language.

$$\Gamma \vdash P \star : \star$$
Subsumption

Many rules are shared.

\[
\Gamma \vdash x : \theta A \in \Gamma \\
\frac{\Gamma \vdash x : A}{\Gamma \vdash \theta x : A} \\
\Gamma \vdash \theta b : \text{Nat} \\
\frac{\Gamma \vdash \theta S b : \text{Nat}}{}
\]

Programmatic language allows features (general recursion, type-in-type, abort etc.) that do not type check in the logical language.

\[
\frac{\Gamma \vdash P *}{\Gamma \vdash P \star : \star}
\]

Logical language is a *sublanguage* of the programmatic language.

\[
\Gamma \vdash^L a : A \\
\frac{\Gamma \vdash^P a : A}{\Gamma \vdash^P a : A}
\]
Mixing proofs and programs

These two languages are not independent.

- Should be able to allow programs to manipulate proofs, and proofs to talk about programs.
- Data structures (in both languages) should have both logical and programmatic components.
New type form $A@\theta$ internalizes the judgement

$$\Gamma \vdash^\theta v : A$$
The @ Modality

New type form $A@\theta$ internalizes the judgement

$$\Gamma \vdash^\theta v : A$$

Introduction form embeds values from one language into the other.

$$\Gamma \vdash^\theta v : A \quad \frac{\Gamma \vdash^\theta v : A}{\Gamma \vdash^\theta' \text{box} v : A@\theta}$$
The @ Modality

New type form $A@\theta$ internalizes the judgement

$$\Gamma \vdash^\theta v : A$$

Introduction form embeds values from one language into the other.

$$\Gamma \vdash^\theta v : A \quad \Rightarrow \quad \Gamma \vdash^\theta \text{box} v : A@\theta$$

Elimination form derived from modal type systems.

$$\Gamma \vdash^\theta a : A@\theta' \quad \Gamma, x : \theta' A, z :_{\bot} \text{box} x = a \vdash^\theta b : B \quad \Gamma \vdash^\theta B : s \quad \Rightarrow \quad \Gamma \vdash^\theta \text{unbox}_z x = a \ \text{in} \ b : B$$
Components of a pair are from the same language by default.

\[
\begin{align*}
\Gamma &\vdash^{\theta} a : A & \Gamma &\vdash^{\theta} b : [a/x]B \\
\Gamma &\vdash^{\theta} [a/x]B : s & \Gamma &\vdash^{\theta} \Sigma x : \forall x : A. B : s \\
\hline
\Gamma &\vdash^{\theta} (a, b) : \Sigma x : A. B
\end{align*}
\]

Programs can embed proofs about data.

\[
\vdash^{P} (0, \text{box } v) : \Sigma x : \forall x : \text{Nat.}(\forall y : \text{Nat.} (x \leq y))@L
\]

Data structures are parametric in their logicality. The same datatype can store a list of proofs as well as a list of program values.
Abstraction

Standard abstraction rule conflicts with subsumption.

\[
\frac{
\Gamma, x : \theta \ A \vdash \theta \ a : B \quad \Gamma \vdash \theta \ (x : A) \to B : s
}{
\Gamma \vdash \theta \ \lambda x. a : (x : A) \to B
}\
\]
Solution

Require every argument type to be an $A@\theta$ type, so subsumption has no effect.

$$
\Gamma, x : \theta' \ A \vdash \theta \ b : B \quad \Gamma \vdash \theta \ (x : \theta' \ A) \to B : s
$$

$$
\Gamma \vdash \theta \ \lambda x. b : (x : \theta' \ A) \to B
$$

Application implicitly boxes.

$$
\Gamma \vdash \theta \ a : (x : \theta' \ A) \to B \quad \Gamma \vdash \theta \ \text{box} \ b : A@\theta' \quad \Gamma \vdash \theta \ [b/x]B : s
$$

$$
\Gamma \vdash \theta \ a \ b : [b/x]B
$$
Logical preconditions

Programmatic functions can have logical parameters:

$$\Gamma \vdash^P \text{div} : (n \ d :^P \text{Nat}) \rightarrow (p :^L d \neq 0) \rightarrow \text{Nat}$$

Such arguments are “proofs” that the preconditions of the function are satisfied.
Logical functions can have programmatic parameters:

$$\Gamma \vdash^L ds : (n \ d :^P \text{Nat}) \rightarrow (p :^L d \neq 0) \rightarrow (\sum z : \text{Nat}. z = \text{div } n \ d)$$
Logical functions can have programmatic parameters:

\[
\Gamma \vdash^L ds : (n \ d :^P \text{Nat}) \to (p :^L d \neq 0) \to (\Sigma z : \text{Nat}. z = \text{div} \ n \ d)
\]

\(ds\) is a proof that \(\text{div}\) terminates for nonzero arguments, even if \(\text{div}\) was originally defined with general recursion.
Some values are shared between the two languages.
Shared values

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- For example, all natural numbers are values in the logical language as well as in the programmatic language.
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- For example, all natural numbers are values in the logical language as well as in the programmatic language.
- This means that it is sound to treat a variable of type `Nat` as logical, no matter what it is assumed to be in the context.

\[
\Gamma \vdash^P v : \text{Nat} \\
\Gamma \vdash^L v : \text{Nat}
\]
Uniform equality

- Equality proofs are also shared.

\[
\begin{align*}
\Gamma & \vdash^p v : A = B \\
\Gamma & \vdash^l v : A = B
\end{align*}
\]

- This supports incremental verification. We can have a partial function return an equality proof and then use its result to satisfy logical preconditions.
Uniform equality

- Equality proofs are also shared.

\[\Gamma \vdash^P v : A = B\]
\[\Gamma \vdash^L v : A = B\]

- This supports incremental verification. We can have a partial function return an equality proof and then use its result to satisfy logical preconditions.

- However, we currently only know how to add this rule to logical languages with *predicative* polymorphism. Girard’s trick interferes.
Challenge: the internalized type.

\[
\Gamma \vdash^P v : A@\theta \\
\Gamma \vdash^L v : A@\theta
\]

This allows proofs embedded in programs to be used when reasoning about those programs (not just as preconditions to other programs).

Promising initial results via step-indexed semantics, limitations necessary.
Related work

- Bar types in Nuprl - no admisibility required
- Partiality Monad
- F-star kinds
- ML5, distributed ML
Future work

- What can we add to the logical language? Large Eliminations?
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- Interaction with classical reasoning: allow proofs to branch on whether a program halts or diverges
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- Interaction with classical reasoning: allow proofs to branch on whether a program halts or diverges
- Elaboration to an annotated language
Summary

- Can have full-spectrum dependently-typed language with nontermination, effects, etc.
- Call-by-value semantics permits “partial correctness”
- Logical and programmatic languages can interact
  - All proofs are programs
  - Logic can talk about programs
  - Programs can contain proofs
  - Some values can be transferred from programs to logic