A Proof-Theoretic Perspective on
Dependently-typed Functional Programming

or: Dependently-typed Functional Programming is Proof Search

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based on joint work with a.o.
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especial debt to Rod Burstall

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DTP talk 2011-09-16 NII/Shonan Village Center
some old (JFP 2004; EPIGRAM 1) and more recent (CSL 2006; LMCS 2011) work

some extensions/generalisations:
- modest extensions to EPIGRAM 1-style intended to reduce premature commitment
- re-designing type theory in sequent calculus style to support postponed decisions

some (open?) questions; stimulus for discussion

Morrisett: pragmatics as the ‘undiscovered country’ for PL researchers
introduction

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dialogue systems are for the interactive construction of a mathematical object with a dependent type (Bengt)

smart case: you don’t want to work with the ‘with’ rule (Thorsten)

extend recursion beyond the non-structural case (Tim)

parametricity: the interpretation of a type is a relation (Patrik)

functional induction: induction on the graph relation is partial correctness for a function definition; ‘Below’ is bad (Matthieu)

you want to turn off the termination checker in Agda (Stephanie)
(intuitionistic) dependent type theory via C-H/de B/M-L" is:

- lots of interesting things... (deleted)
- a very rich syntax for well-orderings
- a functional language for proofs: evidence for typing judgments

\[ \text{hypotheses} \vdash \text{prf} : \text{conclusion} \]

harmony between introduction and elimination yields WN

- a total functional language for programming: evidence for meeting a specification

\[ \text{declarations, definitions} \vdash \text{prog} : \text{specification} \]
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- a total functional language for programming: evidence for *meeting a specification*

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“we know a proof when we see one” (Kreisel)

Fundamental property:
- typing judgment $\Gamma \vdash M : A$ is decidable
- by reduction to type synthesis $\Gamma \vdash M \Rightarrow B$
- and type conversion $\Gamma \vdash B \simeq A$

Idea: to compute $B$, look at structure of $M$!

Modern version: bidirectional typechecking, mixing synthesis and checking $\Gamma \vdash M \Leftarrow A$

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problems-as-types

programming is interactive, type-directed, problem solving
Why Proof Search? How?

Under types as propositions,
- type inhabitation $\Gamma \vdash A \implies M$ corresponds to provability
- existence of a proof of $A$ is... existence of a program
- so programming is (the end result of) searching for proofs

Clearly impossible/undecidable in general, but easy heuristics:
- to inhabit $\Pi$, try $\lambda$ and recur; otherwise
- pick an assumption whose type suitably matches the goal
- recursively search for arguments to supply to yield an application term

For the purely functional fragment:
- Dowek: complete for enumeration of inhabitants
- Dyckhoff/Hudelmaier: terminating, for simple enough types

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interaction/implementation styles: identify your favourite!

- programming = program construction
- program construction = proof term construction (really?)
- proof term construction = ... 
  - but: we tend to think of this as $\lambda$-term construction
    - direct-style (term editing): ALF, Agda, ...
    - indirect-style (tactic scripts): NuPRL, Coq, ...
    - semi-indirect (elaboration): Epigram, Agda (?), Equations...

- ‘Joe Programmer’ (writes it all, machine maybe typechecks it): Idris (Brady), $F^*$ (?), Trellys (?)...

**Question** is this last what people really want?
More serious: how much does the user write? what does the machine supply?
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programming pragmatics/psychology of programming

obstacles to fully-fledged DTP from opposite directions:

- theoretical: desire for an evolutionary path from Hindley-Milner languages
- cognitive: lack of evolutionary path from Hindley-Milner languages

each an entirely understandable cultural conservatism

HCI/PPoP perspective: Green/Blackwell framework of cognitive dimensions of notations
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- Theoretical: desire for an evolutionary path from Hindley-Milner languages.
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Each obstacle is an entirely understandable cultural conservatism.

The HCI/PPoP perspective is the Green/Blackwell framework of cognitive dimensions of notations.
Two views of data and control in programming

Classical view:

- data structures consist of structures containing data
- general recursion/iteration as universal traversal over such structure, exposing the data by repeated computation/case analysis
- termination and even correctness (!), analysed post hoc

“Easier” view:

- data structures consist of data exposing visible (inductive) structure
- primitive (structural) recursion traverses over such structure; no need to expose substructure by computation
- termination “for free”; correctness easier if you choose datatypes carefully
Datatype families and programming with DTs

data:

▶ usual (strictly positive) algebraic datatypes from FP
▶ non-context-free syntax, e.g. well-typed terms
▶ inductively-defined relations (incl. partial functions)
▶ sos definitions of your favourite operational semantics
▶ set(oid) theoretic definitions of algebraic structure, so denotational semantics too

programs:

▶ λ-calculus for contextual/higher-order functional plumbing
▶ case analysis/primitive recursion for well-founded (inductive) data
▶ ... for productive infinite computation on co-inductive data
Inductive families, with declarations

\[
\text{data } \overrightarrow{t} : \overrightarrow{T} \\
D \overrightarrow{t} : \star \\
\text{where } \frac{\Delta_1}{c_1 \Delta_1 : D \overrightarrow{s}_1} \quad \ldots \quad \frac{\Delta_n}{c_n \Delta_n : D \overrightarrow{s}_n}
\]

giving rise to standard Martin-Löf elimination constants \texttt{D-elim} and corresponding \texttt{\iota}-reductions.

Programs are top-level definitions of typed terms in the underlying type theory, but syntax is “high-level”: typechecker fills in many details.
EPIGRAM 1: use the programmer to control search

programmer chooses:

- left-hand sides: ‘case analysis’ \((\Leftarrow)\)
- recursion schemes: identify allowable recursive calls (also \(\Leftarrow!\))
- right-hand sides: solutions to ‘leaf’ problems \((\Rightarrow)\)
- intermediate computation \((\|, \text{not ‘let’ as such})\)

Each amounts to supplying (sufficient) evidence to solve the corresponding problem.

Informal justification by appeal to left-/right-rules in sequent calculus; ‘with’ is cut

**Problem** every program begins with commitment to some rec!

**Question** what is the right syntax for ‘sufficient evidence’?

**Question** what evidence is (run-time) erasable?
Eliminator types: what are allowable recursive calls?

Standard case analysis for family $\mathcal{D}$ always available:

$$
\textbf{D-case} \quad \forall_{\vec{t}} x : \mathcal{D} \vec{t} \rightarrow \forall P : (\forall_{\vec{t}} x : \mathcal{D} \vec{t} \rightarrow \star) \rightarrow \\
\quad \forall m_1 : (\forall \Delta_1. P (c_1 \vec{s}_1)) \rightarrow \ldots \quad \forall m_n : (\forall \Delta_n. P (c_n \vec{s}_n)) \rightarrow P x
$$

while a general form of recursion principle:

$$
\textbf{D-Frec} \quad \forall_{\vec{t}} x : \mathcal{D} \vec{t} \rightarrow \forall P : (\forall_{\vec{t}} x : \mathcal{D} \vec{t} \rightarrow \star) \rightarrow \\
\quad (\forall_{\vec{t}} y : \mathcal{D} \vec{t} \rightarrow F(P) y \rightarrow P y) \rightarrow P x
$$

may be admissible according to the form of $F$. Always have:

**primitive recursion** recursive calls on the immediate subterms

$$
F_{pr}(P)(c_i \vec{s}_i) \simeq \times_j (P s_{ij})
$$

**structurally smaller** recursive calls on all subterms (‘Below’):

$$
F_{ss}(P)(c_i \vec{s}_i) \simeq (\times_j ((F_{ss}(P)) s_{ij})) \times (\times_j (P s_{ij}))
$$

**well-founded** for provably well-founded relations $R$

$$
F_{wf}(P)(y) \simeq \forall z \rightarrow (R z y) \rightarrow P z
$$
The predicate transformer (functional) $F$ describes a container of possible recursive calls $F(P)$ available for a given argument $y$, obtained by lookup. (Bad!)

Allowable recursion/co-recursion given by identifying suitable algebras/co-algebras for such functors (Uustalu, Capretta, Vene); modern treatments of data/codata systematically go via (indexed) containers (Thorsten et al.)

**Question**: is there a compositional account of such functors?

**Question**: is there a convenient syntax for such things?

**Extension**: require outermost appeal to $F_{rec}$, but delay choice of $F$
Other sources of premature commitment

- functional induction: graph of a higher-order function is a predicate transformer (cf. parametricity), and the proof that the function inhabits the graph is a proof transformer
- elimination with a motive: not necessarily with respect to \textit{equality}, but with respect to an arbitrary \textit{reflexive} $R$ which reflects congruences for appropriate constructors (e.g. permutation on lists)
- equality elimination/substitutivity in a type $P x$: not necessarily with respect to \textit{equality}, but with respect to some $R$ for which given $P$ is ‘good’ (cf. setoid rewriting)
- identify your favourites!
- failure of syntax-directedness leading to smart case?
- computational behaviour of programs defined by Equations; corresponding choices in EPIGRAM 1, Idris

However, deferral imposes different heavy cognitive burden
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Part II: proof search in type theory

Classical approach to premature commitment in proof search in natural deduction (NJ): use sequent calculus (LJ)

- source of premature commitment: choice of antecedent formula in $\rightarrow$-elim
- solution: left-/right rules (LJ), rather than intro-/elim- rules (NJ)
- a calculus for inhabitation of corresponding NJ formulas-as-types
- unification/meta-variables delay choice of term witnesses to $\forall$-left instances

Lots of literature, esp. now on extensions to dependent types

Almost none on using this for programming (Wadler, 1990s, unpublished)
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For arbitrary PTSs, can develop a term calculus with two judgment forms:

- \( \Gamma \vdash M : A \) corresponding to \( \Gamma \vdash A \gg M \)
- \( \Gamma ; A \vdash l : B \) corresponding to computing argument lists to “match” \( A \) against \( B \)

Key idea: LJ is too permissive, so tighten up to remove inessential variation (permutation of rules)

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Adding meta-variables (LMCS 2011)

leads to a calculus in which

- Dowek’s complete semi-recursive type inhabitation procedure can be recovered, hence higher-order unification
- Dyckhoff/Hudelmaier complete search for propositional sub-languages

Challenge extend analysis to datatypes, thereby

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- modernising, to deal with e.g. bidirectional type checking, ...
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Advantages for the implementor?

Such calculi combine
- explicit substitutions
- spine representations
so hopefully better adapted towards
- abstract machines for evaluation
- ‘internal’ (inferential mode) and ‘external’ (checking mode) categories of abstract syntax in recent presentations of \textsc{Epigram} 2 (Chapman, Alti, McBride . . .)

Metavariables and unification/conversion are baked in from the start, so there is no separate ‘program construction’ layer distinct from that of eventually elaborated programs: these are just terms containing no open meta-variables.
Rules, I

\[
\Gamma \vdash_{PE} M : A \mid \Sigma
\]

\[
C \xrightarrow{\ast_{Bx}} s \quad (s', s) \in \mathcal{A}
\]

\[
\Gamma \vdash_{PE} s' : C \mid \emptyset
\]

\[
\text{sorted}
\]

\[
C \xrightarrow{\ast_{Bx}} s \quad (s_1, s_2, s) \in \mathcal{R}
\]

\[
\Gamma \vdash_{PE} A : s_1 \mid \Sigma_1 \quad \Gamma, x : A \vdash_{PE} B : s_2 \mid \Sigma_2
\]

\[
\Gamma \vdash_{PE} \Pi x^A . B : C \mid \Sigma_1, \Sigma_2
\]

\[
(x : A) \in \Gamma \quad \Gamma \vdash_{PE} l : C \mid \Sigma
\]

\[
\Gamma \vdash_{PE} x \mid l : C \mid \Sigma
\]

\[
\text{Select}_x
\]

\[
C \xrightarrow{\ast_{Bx}} \Pi x^A . B \quad \Gamma, x : A \vdash_{PE} M : B \mid \Sigma
\]

\[
\Gamma \vdash_{PE} \lambda x^A . M : C \mid \Sigma
\]

\[
\text{\Pi} r
\]
\[ \Gamma; B \vdash_{PE} I : C \mid \Sigma \]

\[ \Gamma = x_1 : A_1, \ldots, x_n : A_n \]

\[ \Gamma \vdash_{PE} \alpha(x_1 [], \ldots, x_n []) : C \mid (\Gamma \vdash \alpha : C) \quad \text{Claim}_\alpha \]

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\[ \Gamma; D \vdash_{PE} \beta(x_1 [], \ldots, x_n []) : C \mid (\Gamma; D \vdash \beta : C) \quad \text{Claim}_\beta \]

\[ \Gamma; D \vdash_{PE} [] : C \mid D = C \quad \text{axiom} \]

\[ D \xrightarrow{*}_{Bx} \Pi x^A . B \quad \Gamma \vdash_{PE} M : A \mid \Sigma_1 \quad \Gamma; \langle M/x \rangle B \vdash_{PE} I : C \mid \Sigma_2 \quad \Pi \]

\[ \Gamma; D \vdash_{PE} M \cdot l : C \mid \Sigma_1, \Sigma_2 \]
\[ \Sigma \Rightarrow_{PE} \sigma \]

\[ \Gamma; B \vdash_{PE} l : C \mid \Sigma'' \quad \Sigma, \Sigma'', (\beta \mapsto \text{Dom}(\Gamma).l)(\Sigma') \Rightarrow_{PE} \sigma_{\Sigma}, \sigma_{\Sigma''}, \sigma_{\Sigma'} \]

\[ \Sigma, (\Gamma; B \vdash \beta : C), \Sigma' \Rightarrow_{PE} \sigma_{\Sigma}, (\beta \mapsto \text{Dom}(\Gamma).(\sigma_{\Sigma}, \sigma_{\Sigma''})(l)), \sigma_{\Sigma'} \]

\[ \Gamma \vdash_{PE} M : A \mid \Sigma'' \quad \Sigma, \Sigma'', (\alpha \mapsto \text{Dom}(\Gamma).M)(\Sigma') \Rightarrow_{PE} \sigma_{\Sigma}, \sigma_{\Sigma''}, \sigma_{\Sigma'} \]

\[ \Sigma, (\Gamma \vdash \alpha : A), \Sigma' \Rightarrow_{PE} \sigma_{\Sigma}, (\alpha \mapsto \text{Dom}(\Gamma).(\sigma_{\Sigma}, \sigma_{\Sigma''})(M)), \sigma_{\Sigma'} \]

\[ \Sigma \text{ is solved} \]

\[ \Sigma \Rightarrow_{PE} \emptyset \text{ Solved} \]
Conclusions

▷ dependent type theory as a nice place to study correct-by-construction programming
▷ ... which is type-directed, interactive, proof search
▷ machinery for type-checking/type synthesis/conversion testing modulo unknowns
▷ unification as a pervasive technology from traditional proof search
▷ many (?) more places during construction when unknowns allow progress without over-committing the programmer
▷ outstanding problem: high-level syntax for sufficient evidence to yield well-typed terms in the underlying theory
▷ outstanding disadvantage: we make the programmer supply (nearly) everything
▷ no treatment yet of undo
Questions?