Parameterized AC^0 – Some upper and lower bounds

Yijia Chen
Fudan University

September 19^{th}, 2017 @ Shonan

AC^0

A family of Boolean circuits $\left(\mathsf{C}_n\right)_{n\in\mathbb{N}}$ are AC^0-circuits if for every $n \in \mathbb{N}$

(i) C_n computes a Boolean function from $\{0,1\}^n$ to $\{0,1\}$;

(ii) the depth of C_n is bounded by a fixed constant;

(iii) the size of C_n is polynomially bounded in n.

Remark

1. *Without (ii), we get a family of polynomial-size circuits* $\left(C_n\right)_{n\in\mathbb{N}}$*, which decides a language in* P/poly.
2. *If* C_n *is computable by a* TM *in time* $O(\log n)$*, then* $\left(\mathsf{C}_n\right)_{n\in\mathbb{N}}$ *is* dlogtime-uniform*, which corresponds to* $\mathsf{FO}(<,+,\times)$ [Barrington, Immerman, and Straubing, 1990].

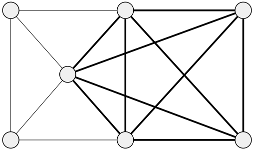
The k-clique problem

Definition

Let G be a graph and $k \in \mathbb{N}$. Then a subset $C \subseteq V(G)$ is a k-clique if

(i) for every two vertices $u, v \in V(G)$ either $u = v$ or $\{u, v\} \in E(G)$,

(ii) and $|C| = k$.



A graph with a 5-clique.

k-clique by AC^0

Let $n \in \mathbb{N}$ and we encode a graph G with $V(G) = [n]$ as follows. For every $1 \leqslant i < j \leqslant n$ the Boolean variable $X_{\{i,j\}}$ is defined by

$$X_{\{i,j\}} = \begin{cases} 1 & \text{if there is an edge between } i \text{ and } j \\ 0 & \text{otherwise.} \end{cases}$$

Then the k-clique problem can be computed by circuits

$$\mathsf{C}_{\binom{n}{2}} = \bigvee_{K \in \binom{[n]}{k}} \bigwedge_{\{i,j\} \in \binom{K}{2}} X_{\{i,j\}}.$$

(i) $\mathsf{C}_{\binom{n}{2}} : \{0,1\}^{\binom{n}{2}} \to \{0,1\}$.

(ii) The depth of $\mathsf{C}_{\binom{n}{2}}$ is 2.

(iii) $\mathsf{C}_{\binom{n}{2}}$ has size $n^{k+O(1)}$.

Rossman's Theorem

Theorem (Rossman, 2008)

Let $k \in \mathbb{N}$. There are no AC^0-circuits $\left(\mathsf{C}_{\binom{n}{2}}\right)_{n\in\mathbb{N}}$ of size $O(n^{k/4})$ such that for every n-vertex graph G

$$G \text{ has a } k\text{-clique} \iff \mathsf{C}_{\binom{n}{2}}(G) = 1.$$

A uniform version

Corollary

There are no circuits $\left(\mathsf{C}_{\binom{n}{2},k}\right)_{n,k\in\mathbb{N}}$ *which satisfy the following conditions.*

(i) *The size of* $\mathsf{C}_{\binom{n}{2},k}$ *is bounded by* $f(k)\cdot n^{k/4}$*.*

(ii) *The depth of* $\mathsf{C}_{\binom{n}{2},k}$ *is bounded by* $g(k)$*.*

(iii) *Let* G *be an* n*-vertex graph* G *and* $k\in\mathbb{N}$*. Then*

$$G \text{ has a } k\text{-clique} \iff \mathsf{C}_{\binom{n}{2}}(G)=1.$$

Remark

1. *It is about the circuit complexity of the parameterized clique problem.*
2. *If true without (ii), then the parameterized clique problem is not fixed-parameter tractable. Thus it is an* AC^0 *version of* $\mathsf{FPT}\neq\mathsf{W}[1]$*.*

Outline

1. Parameterized AC^0
2. Some lower bounds
   - for fpt-approximation of the clique problem.
3. Some upper bounds:
   - a descriptive characterizations of parameterized AC^0,
   - the color coding technique in parameterized AC^0.

Parameterized AC^0

Parameterized problems

Definition

A parameterized problem (Q, κ) consists of a classical problem $Q \subseteq \Sigma^*$ and a function $\kappa : \Sigma^* \to \mathbb{N}$, the parameterization, computable in polynomial time.

Example

p-CLIQUE	
Input:	A graph G and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Does G contain a clique of size k?

p-DOMINATING-SET	
Input:	A graph G and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Does G contain a dominating set of size k?

Parameterized AC^0

Definition (Bannach, Stockhusen, and Tantau, 2015)

A parameterized problem (Q, κ) is in para-AC^0 if there exists a family $\left(C_{n,k}\right)_{n,k\in\mathbb{N}}$ of circuits such that:

1. The depth of every $C_{n,k}$ is bounded by a fixed constant.
2. $|C_{n,k}| \leqslant f(k) \cdot n^{O(1)}$ for every $n, k \in \mathbb{N}$.
3. Let $x \in \Sigma^*$. Then ($x \in Q$ if and only if $C_{|x|,\kappa(x)}(x) = 1$).
4. There is a TM that on input $(1^n, 1^k)$ computes the circuit $C_{n,k}$ in time $g(k) + O(\log n)$.

Both $f, g : \mathbb{N} \to \mathbb{N}$ are computable functions.

Some equivalent characterizations

Proposition

Let (Q, κ) *be a parameterized problem with* κ *computable by* AC^0*-circuits. Then all the following are equivalent.*

(i) $(Q, \kappa) \in$ para-AC^0.

(ii) [AC^0 after a precomputation] *There is a computable function* $pre : \mathbb{N} \to \Sigma^*$ *and dlogtime-uniform* AC^0*-circuits* $(\mathsf{C}_n)_{n\in\mathbb{N}}$ *such that for* $x \in \Sigma^*$,

$$x \in Q \iff \mathsf{C}_{|(x, pre(\kappa(x)))|}(x, pre(\kappa(x))) = 1.$$

(iii) [Eventually in AC^0] Q *is decidable and there is a computable function* $h : \mathbb{N} \to \mathbb{N}$ *and dlogtime-uniform* AC^0*-circuits* $\left(\mathsf{C}_n\right)_{n\in\mathbb{N}}$ *such that for every* $x \in \Sigma^*$ *with* $|x| \geq h(\kappa(x))$,

$$x \in Q \iff C_{|x|}(x) = 1.$$

Some Lower Bounds

Theorem (Rossman, 2008)

p-CLIQUE $\notin$ para-AC^0.

By appropriate reductions, i.e., para-AC^0-reductions:

Corollary

1. p-DOMINATING-SET $\notin$ para-AC^0, *an* AC^0 *version of* FPT $\neq$ W[2].
2. p-WSAT$(\Gamma_{t,d})$ $\notin$ para-AC^0 *for* $t + d \geq 3$, *an* AC^0 *version of* FPT $\neq$ W[t].

Inapproximability of p-Clique by para-AC^0

A major open problem in parameterized complexity

Can we approximate p-CLIQUE in fpt time?

Approximation of p-CLIQUE

Let $\rho : \mathbb{N} \to \mathbb{R}_{\geq 1}$ be a computable function with nondecreasing and unbounded $k \mapsto k/\rho(k)$.

Definition

An algorithm $\mathbb{A}$ is a parameterized approximation for p-CLIQUE with approximation ratio ρ if for every graph G and $k \in \mathbb{N}$ with $\omega(G) \geq k$ the algorithm $\mathbb{A}$ computes a clique C of G such that $|C| \geq k/\rho(k)$.

$\omega(G)$ is the size of a maximum clique of G.

Conjecture

p-CLIQUE *has no parameterized approximation for any* ρ.

Theorem (Chalermsook, Cygany, Kortsarz, Laekhanukit, Manurangsi, Nanongkai, and Trevisan, 2017)

Under the gap Exponential Time Hypothesis, p-CLIQUE has no parameterized approximation for any ρ.

Remark

The gap Exponential Time Hypothesis might require the construction of linear PCP, which seems to be out of reach at this point.

Approximation in para-AC^0

p-$\textsc{Gap}_\rho$-$\textsc{Clique}$	
Input:	A graph G and $k \in \mathbb{N}$ such that either $k \leqslant \omega(G)/\rho(\omega(G))$ or $k > \omega(G)$.
Parameter:	k.
Problem:	Is $k \leqslant \omega(G)/\rho(\omega(G))$?

Lemma

If p-$\textsc{Gap}_\rho$-$\textsc{Clique} \notin$ FPT, then p-$\textsc{Clique}$ has no parameterized approximation with ratio ρ.

Theorem (*C.* and Flum, 2016)

p-$\textsc{Gap}_\rho$-$\textsc{Clique} \notin$ para-AC^0 *for any ρ.*

The proof is based on an AC^0 version of the planted clique conjecture with respect to Erdős-Rényi random graphs.

Erdős-Rényi random graphs

Definition

Let $n \in \mathbb{N}$ and $p \in \mathbb{R}$ with $0 \leqslant p \leqslant 1$. Then $G \in \mathrm{ER}(n, p)$ is the Erdős-Rényi random graph on vertex set $[n]$ constructed by adding every edge $e \in \binom{[n]}{2}$ independently with probability p.

Example

$\mathrm{ER}(n, 1/2)$ is the uniform distribution on graphs with vertex set $[n]$.

Let $G \in \mathrm{ER}(n, 1/2)$. Then the expected $\omega(G)$ is approximately $2 \cdot \log n$.

Erdős-Rényi random graphs with a planted clique

Definition

Let $n \in \mathbb{N}$ and $p \in \mathbb{R}$ with $0 \leqslant p \leqslant 1$. Moreover let $c \in [n]$. Then $(G + A) \in \mathrm{ER}(n, p, c)$ is the distribution:

1. Pick $G \in \mathrm{ER}(n, p)$.
2. Pick a uniformly random subset $A \subseteq [n]$ with $|A| = c$.
3. Plant in G a clique $C(A)$ on A, thus getting the graph $G + C(A)$.

Example

With high probability, the maximum clique in $G + C(A)$ with

$$(G + A) \in \mathrm{ER}(n, 1/2, 4 \cdot \log n)$$

is the clique $C(A)$.

The planted clique conjecture

Conjecture (Jerrum, 1992; Kucera, 1995)

For every polynomial time algorithm $\mathbb{A}$ and for all sufficiently large $n \in \mathbb{N}$

$$\Pr_{(G+A)\in \mathrm{ER}(n,1/2,4\cdot\log n)}\Big[\mathbb{A}\big(G + C(A)\big) \neq A\Big] > \frac{1}{2}.$$

That is, $\mathbb{A}$ fails to find the planted clique with high probability.

An AC^0 version of the planted clique conjecture

Theorem (*C.* and Flum, 2016)

Let $k : \mathbb{N} \to \mathbb{R}^+$ *with* $\lim_{n\to\infty} k(n) = \infty$, *and* $c : \mathbb{N} \to \mathbb{N}$ *with* $c(n) \leqslant n^{\xi}$ *for some* $0 \leqslant \xi < 1$. *Then for all* AC^0*-circuits* $(\mathsf{C}_n)_{n\in\mathbb{N}}$,

$$\lim_{n\to\infty} \Pr_{(G,A)\in \mathsf{ER}(n,n^{-1/k(n)},\ c(n))} \Big[\mathsf{C}_n(G) = \mathsf{C}_n\big(G + C(A)\big) \Big] = 1.$$

Let $(G, A) \in \mathsf{ER}(n, n^{-1/k(n)},\ c(n))$, then

$$\frac{\omega(G + C(A))}{\omega(G)}$$

can be arbitrarily large. Hence

Theorem (*C.* and Flum, 2016)

p-$\textsc{Gap}_\rho$-$\textsc{Clique} \notin$ para-AC^0.

Inapproximability of p-DOMINATING-SET by para-AC^0

Theorem (*C.* and Lin, 2017)

p-GAP$_\rho$-DOMINATING-SET $\notin$ para-AC^0 for

$$\rho(k) = \frac{\log k}{\omega(\log \log k)}.$$

FPT $\setminus$ para-$AC^0 \neq \emptyset$

p-stConn	
Input:	A graph G, $s, t \in V(G)$, and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Does G contain a path from s to t of length $\leqslant k$?

Theorem (Beame, Impagliazzo, and Pitassi, 1995)

p-stConn is not in parameterized AC^0, *even on graphs of degree at most 2.*

Some Upper Bounds

p-VERTEX-COVER	
Input:	A graph G and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Does G contain a vertex cover of size k?

Theorem (Bannach, Stockhusen, and Tantau, 2015)

p-VERTEX-COVER *is in parameterized* AC^0*.*

Remark

The proof of Bannach et al. is direct by circuits, which can be rephrased in first-order logic by a descriptive characterization of para-AC^0*.*

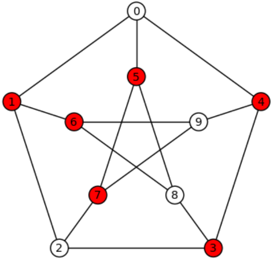
The k-vertex-cover problem

Definition

Let G be a graph and $k \in \mathbb{N}$. Then a subset $C \subseteq V(G)$ is a k-vertex-cover if

(i) for every edge $\{u, v\} \in V(G)$ either $u \in C$ or $v \in C$,

(ii) and $|C| = k$.



The peterson graph with a 6-vertex-cover.

k-vertex-cover by FO

$$G \text{ has a } k\text{-vertex-cover} \iff G \models \psi_k$$

$$\text{where } \psi_k = \exists x_1 \cdots \exists x_k \Big(\bigwedge_{1 \leqslant i < j \leqslant k} x_i \neq x_j \\ \wedge \forall u \forall v \big(Euv \to \bigvee_{i \in [k]} (u = x_i \vee v = x_i) \big) \Big).$$

Can we do better?

Better in what sense?

The quantifier rank of

$$\psi_k = \exists x_1 \cdots \exists x_k \Big(\bigwedge_{1 \leqslant i < j \leqslant k} x_i \neq x_j \\ \wedge \forall u \forall v \big(Euv \to \bigvee_{i \in [k]} (u = x_i \vee v = x_i) \big) \Big)$$

is $\mathrm{qr}(\psi_k) = k + 2$.

There is an algorithm which checks whether

$$\mathcal{A} \models \varphi$$

in time $O(|\varphi| \cdot \|\mathcal{A}\|^{\mathrm{qr}(\varphi)})$.

Definition

Let $q \in \mathbb{N}$. Then FO_q is the fragment of FO consisting of all formulas of quantifier rank at most q.

By simple Ehrenfeucht-Fraïssé games

Theorem

There is no $\varphi \in \mathrm{FO}_{k-1}$ such that for every graph G

$$G \text{ has a } k\text{-vertex cover} \iff G \models \varphi.$$

With arithmetics

Theorem (*C.* , Flum, and Huang, 2017)

For every $k \in \mathbb{N}$ *there is a* $\psi_k \in \mathsf{FO}_{17}$ *such that for every graph* G

$$G \text{ has a } k\text{-vertex cover} \iff (G, <, +, \times, \mathbf{0}, \ldots, \mathbf{k'}) \models \psi_k.$$

Moreover, the mapping is

$$k \mapsto \psi_k$$

is computable (hence, so is $k \mapsto k'$*).*

The slicewise definability of the vertex cover problem

Theorem

p-VERTEX-COVER *is slicewise definable in* FO_{17}. *That is, for every* $k \in \mathbb{N}$, *the* k*th slice of* VERTEX-COVER *i.e., the* k*-vertex-cover problem, is definable by some* $\psi_k \in \mathsf{FO}_{17}$.

Moreover, $k \mapsto \varphi_k$ *is computable.*

The descriptive characterization of para-AC^0

Theorem (*C.* , Flum, and Huang, 2017)

Let (Q, κ) be a parameterized problem. Then (Q, κ) is slicewise definable in FO_q for some $q \in \mathbb{N}$ if and only if $(Q, \kappa) \in$ para-AC^0.

The main theorem

Theorem

p-VERTEX-COVER *is slicewise definable in* FO_{17}.

The proof strategy

1. There is a polynomial time algorithm $\mathbb{K}$ which for every graph G and $k \in \mathbb{N}$ computes a graph G' and k' such that
   1.1 G has k-vertex-cover if and only if G' has a k'-vertex-cover.
   1.2 $|V(G')| \leqslant k^2 + k$ and $k' \leqslant k$.

   $\mathbb{K}$ is known as Buss' kernelization of $\textsc{Vertex-Cover}$.

2. We show that $\mathbb{K}$ can be implemented in FO_{17}.

3. Any class of graphs with at most $k^2 + k$ vertices can be defined in FO_0 with the constants $\mathbf{0}, \dots, \mathbf{k^2} + \mathbf{k}$.

Buss' kernelization

1. If v is a vertex of degree at least $k+1$, then v must be in every k-vertex cover. Thus we can remove all such v and decrease k accordingly.
2. Remove all isolated vertices.
3. Let G' and k' be the resulting instance. If

   $$|V(G')| > k^2 + k \geq k'(k+1),$$

   then G', and hence also G, is a no instance.

Implementing Buss' kernelization in FO_{17}?

The main difficulty is how to count in FO_{17}, e.g. how to identify a vertex v with degree at least $k+1$.

$$\exists x_1 \cdots \exists x_{k+1} \left(\bigwedge_{1 \leqslant i < j \leqslant k+1} x_i \neq x_j \wedge \bigwedge_{i \in [k]} Evx_i \right)$$

would not work.

Color coding

Lemma (Alon, Yuster, and Zwick, 1995)

For every sufficiently large $n \in \mathbb{N}$, it holds that for all $k \leqslant n$ and for every k-element subset X of $[n]$, there exists a prime $p < k^2 \cdot \log_2 n$ and $q < p$ such that the function $h_{p,q} : [n] \to \{0, \ldots, k^2 - 1\}$ given by

$$h_{p,q}(m) := (q \cdot m \mod p) \mod k^2$$

is injective on X.

Color coding in FO_q

Corollary

Let $k \in \mathbb{N}$ and $\varphi(\bar{x}, y)$ be an FO*-formula. Then there exists an* FO*-formula $\chi_{\varphi,k}(\bar{x})$ of the form*

$$\rho \vee \exists p \exists q \left(\bigvee_{0 \leqslant i_1 < \cdots < i_k < k^2} \bigwedge_{j \in [k]} \exists y \left(\text{“}h_{p,q}(y) = i_j\text{”} \wedge \varphi(\bar{x}, y) \right) \right),$$

such that

1. *for every graph G and $\bar{u} \in V(G)^{|\bar{x}|}$ there are k vertices v in G satisfying $\varphi(\bar{u}, v)$ if and only if*

$$(G, <, +, \times, \mathbf{0}, \ldots, \mathbf{k^3}) \models \chi_{\varphi,k}(\bar{u}),$$

2. *and* $\mathrm{qr}(\chi_{\varphi,k}) = \max\{12, \mathrm{qr}(\varphi) + 3\}$.

Degree constraints by color coding

Let

$$\varphi(x, y) = Exy.$$

Then for every $k \in \mathbb{N}$, every graph G and $v \in V(G)$

$$(G, <, +, \times, \mathbf{0}, \ldots, \mathbf{k^3}) \models \chi_{\varphi,k}(v) \iff \text{the degree of } v \text{ in } G \text{ is at least } k.$$

Moreover, $\mathrm{qr}(\chi_{\varphi,k}) = 12$.

Recall Buss' kernelization

1. If v is a vertex of degree $\geq k + 1$, then v must be in every k-vertex cover. Thus we can remove all such v and decrease k accordingly.
2. Remove all isolated vertices.
3. Let G' and k' be the resulting instance. If $|V(G')| > k^2 + k \geq k'(k + 1)$, then G', and hence also G, is a no instance.

Buss' kernelization in FO_{17}

Corollary

For every $k \in \mathbb{N}$ and $k' \leqslant k$ there are

$$\varphi_{vertex}(x), \varphi_{edge}(x, y), \varphi_{kernel} \in FO_{17},$$

such that for every graph G if we define G' with

$$V(G') := \{v \in V(G) \mid G \models \varphi_{vertex}(v)\},$$
$$E(G') := \{\{u, v\} \mid u, v \in V(G'), u \neq v, \text{and } G \models \varphi_{edge}(u, v)\},$$

then

$$G \text{ has a } k\text{-vertex-cover}$$
$$\iff (G, <, +, \times, \mathbf{0}, \ldots) \models \varphi_{kernel} \text{ and } G' \text{ has a } k'\text{-vertex-cover.}$$

Moreover, $|V(G')| \leqslant k^2 + k$ if $(G, <, +, \times, \mathbf{0}, \ldots) \models \varphi_{kernel}$.

The final step

Lemma

Let H be a graph with $|V(H)| = k$. Then there is an FO_0*-sentence φ_H such that for every graph G*

$$G \text{ and } H \text{ are isomorphic} \iff (G, \mathbf{0}, \ldots, \mathbf{k}) \models \varphi_H.$$

Corollary

Let K be a finite class of graphs closed under isomorphisms. Then there is an FO_0*-sentence φ_K such that for every graph G*

$$G \in K \iff (G, \mathbf{0}, \ldots) \models \varphi_K.$$

Recall:

Corollary

For every $k \in \mathbb{N}$ *and* $k' \leqslant k$ *there are* $\varphi_{vertex}(x), \varphi_{edge}(x, y), \varphi_{kernel} \in \mathrm{FO}_{17}$*, such that for every graph* G *if we define* G' *with*

$$V(G') := \{v \in V(G) \mid G \models \varphi_{vertex}(v)\},$$
$$E(G') := \{\{u, v\} \mid u, v \in V(G'), u \neq v, \text{and } G \models \varphi_{edge}(u, v)\},$$

then

$$G \text{ has a } k\text{-vertex-cover} \iff G \models \varphi_{kernel} \text{ and } G' \text{ has a } k'\text{-vertex-cover}$$

Moreover, $|V(G')| \leqslant k^2 + k$ *if* $G \models \varphi_{kernel}$*.*

But how to define $\mathbf{0}, \mathbf{1}, \ldots$ of G' in G?

Final step by color-coding

What we really need to define a finite graph is to say, e.g.,

there is an edge between the first and the twelfth vertices.

So if we know the subgraph G' of G constructed by Buss' kernelzation, and its size ℓ, then for some p and q, and $0 \leqslant i_1 < \cdots < i_\ell < \ell^2$ we have

$$h_{p,q}(V(G')) = \{i_1, \ldots, i_\ell\}.$$

Thus we can say, the first, the second, ..., vertices in G' in FO_{17}.

Hitting set problems with bounded hyperedge size

d-Hitting-Set		
	Input:	A hypergraph H in which every hyperedge has size at most d and $k \in \mathbb{N}$.
	Problem:	Does G contain a vertex set of size at most k such that it intersects every hyperedge?

Theorem

Let $d \in \mathbb{N}$. Then d-Hitting-Set is slicewise definable in FO_q with $q = O(d^2)$.

What makes vertex-cover/d-hitting-set slicewise definable?

Let X be a set variable. Then p-VERTEX-COVER is Fagin-defined by

$$\varphi(X) := \forall x \forall y \big(\neg Exy \vee Xx \vee Xy\big).$$

More precisely, for every graph G and $S \subseteq V(G)$

$$S \text{ is a vertex cover of } G \quad \Longleftrightarrow \quad G \models \varphi(S).$$

p-CLIQUE is Fagin-defined by

$$\forall x \forall y \big(x = y \vee Exy \vee \neg Xx \vee \neg Xy\big).$$

It is not slicewise definable in any FO_q [Rossman, 2008].

p-DOMINATING-SET is Fagin-defined by

$$\forall x \exists y \big(Xy \wedge (x = y \vee Exy)\big).$$

It is not slicewise definable in any FO_q [*C.* and Flum, 2016].

A meta-theorem

Theorem (*C.* , Flum, and Huang, 2017)

Let $\varphi(X)$ be a formula in which the set variable X does not occur in the scope of an existential quantifier or negation symbol. Then the problem Fagin-defined by $\varphi(X)$ is slicewise definable in FO_q, where q only depends on φ.

Another meta-theorem

Theorem (*C.* and Flum, 2017)

Let **K** *be a class of graphs of bounded tree depth. Then*

p-MC(**K**, MSO)	
Input:	A graph $G \in$ **K** and $\varphi \in$ MSO.
Parameter:	$\lvert\varphi\rvert$.
Problem:	Decide whether $G \models \varphi$.

is in para-AC^0.

If **K** *has unbounded tree depth, and is closed under subgraphs, then* p-MC(**K**, FO) $\notin$ para-AC^0.

Some lower bounds

Building on [Håstad, 1988],

Theorem (*C.* , Flum, and Huang, 1988)

Let $q \in \mathbb{N}$. Then there is a problem slicewise definable in FO_{q+1} *but not in* FO_q*.*

Conclusions

1. As in classical AC^0-complexity, we can prove many unconditional para-AC^0 lower bounds. They might increase our confidence in those parameterized complexity assumptions.
2. Proving classical AC^0-lower bounds likely leads to lower bounds for para-AC^0. Conversely, proving lower bounds for para-AC^0 might require proving optimal AC^0-lower bounds.
3. Can we go beyond AC^0, e.g., circuits with modular counting gates?