Logics with Invariantly Used Relations

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Order-Invariance

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A formula φ is order-invariant if its truth value is the same whenever $<$ is interpreted by a linear order, regardless of which order is chosen.

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A formula φ is order-invariant if its truth value is the same whenever $<$ is interpreted by a linear order, regardless of which order is chosen.
More exactly:

- Let A be a τ-structure. φ over the signature $\tau \cup \{<\}$ is order-invariant on A if

$$(A, L_1) \models \varphi \qquad \Leftrightarrow \qquad (A, L_2) \models \varphi$$

  for all linear orders L_1, L_2 on A.
- φ is order-invariant on a class $\mathcal{C}$ of τ-structures if it is order-invariant on all $A \in \mathcal{C}$.

Order-Invariant Logics

We define

$$<\text{-inv-FO} := \{\varphi \in \mathrm{FO} \mid \varphi \text{ order-invariant on all finite structures}\}$$

- similarly: $<$-inv-MSO, ...
- similarly: successor-invariant logics (succ-inv-FO, succ-inv-MSO, ...)

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- note: capturing sometimes gives order-invariant sentences

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E, Elberfeld, Harwath; MFCS2014

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$<$-invariance of FO undecidable on star forests

- no structural induction,
  no Feferman-Vaught Theorem,
  no Ehrenfeucht-Fraïsee-Games

Part I: Expressive Power

Expressive Power

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All separating examples for FO are graph-theoretically complex

Expressive Power on Restricted Classes of Structures

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q-Equivalent Orderability

Call two structures A and B q-equivalently orderable (written $A \sim_{q,<} B$) if there are linear orders L_A and L_B such that

$$(A, L_A) \equiv_q (B, L_B)$$

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- $\sim_{q,<}$ not in transitive, transitive closure $\approx_{q,<}$ is equivalence relation
- $\approx_{q,<}$ is stronger than $<$-inv-FO $- \equiv$

Blueprint for Collapse Results

Main Tool

Suppose for every q there is a q' such that

$$A \equiv_{q'} B \qquad \Rightarrow \qquad A \sim_{q,<} B$$

for all $A, B \in \mathcal{C}$. Then $<$-inv-FO $\equiv$ FO on $\mathcal{C}$.

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- Collapse is shown even if invariance is assumed on $\mathcal{C}$ (rather than all finite structures)
- $\sim_{q,<}$ can be shown with EF-games on suitably ordered pairs of structures
- still, getting winning strategies only from $A \equiv_{q'} B$ is too complicated

Benedikt, Segoufin; CSL 2005, Niemisto 2005

<-inv-FO $\equiv$ FO on

- unsiblinged trees (ranked and unranked)
- siblinged ranked trees

- proof uses algebraic tree language theory
- main tool: if $S \equiv_{q'} T$ there is a sequence

$$S = T_0 \sim T_1 \sim \cdots \sim T_\ell = T,$$

where $A \sim B$ if B is obtained from A by pumping or swapping subtrees.

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- also $<$-inv-FO $\equiv$ FO on siblinged unranked trees of bounded depth (Dawar, E; unpublished)

Bounded Tree-Depth

E, Elberfeld, Harwath, MFCS 2014

On graphs of tree-depth at most d:

$\phi \in$	$<$-inv-FO	MSO	$<$-inv-MSO
$\psi \in$	FO	FO	FO+MOD
$\|\psi\|$	$(2d+1)$-exp(q)	$\mathcal{O}(d^2)$-exp(q)	non-elementary
qad(ψ)	$\mathcal{O}(d)$	$\mathcal{O}(d)$	$\mathcal{O}(d)$

- bounded depth tree-decomposition definable in FO
- collapse results follow by Benedikt/Segoufin
- to get succintness: define FO-type of a canonically ordered expansion

Further Results

Elberfeld, Frickenschmidt, Grohe; LICS2016

On classes of graphs of bounded tree-width and classes of graphs excluding some $K_{3,\ell}$ as a minor, the following hold:

$$<\text{-inv-MSO} \equiv \text{CMSO} \qquad \text{and} \qquad <\text{-inv-FO} \preceq \text{MSO}$$

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Grohe, Schwentick; MFCS 1998

Gaifman-locality for $<$-inv-FO (but no Gaifman normal form)

Anderson, van Melkebeek, Schweikardt, Segoufin; ICALP 2011

Arb-invariant FO has polylogarithmic locality radius.

Part II: Model Checking

Complexity of Model Checking

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$$f(|\varphi|) \cdot |G|^c$$

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  for some $f : \mathbb{N} \to \mathbb{N}$ and $c \in \mathbb{N}$
  (*fixed-parameter tractability, fpt*)
- brute force for FO gives

  $$O(|V|^{\mathrm{qr}(\varphi)}),$$

  which is not fpt
- for MSO, even checking $G \models \varphi_{3-\mathrm{col}}$ is NP-complete
  ($\leadsto$ fpt on arbitrary graphs unlikely)

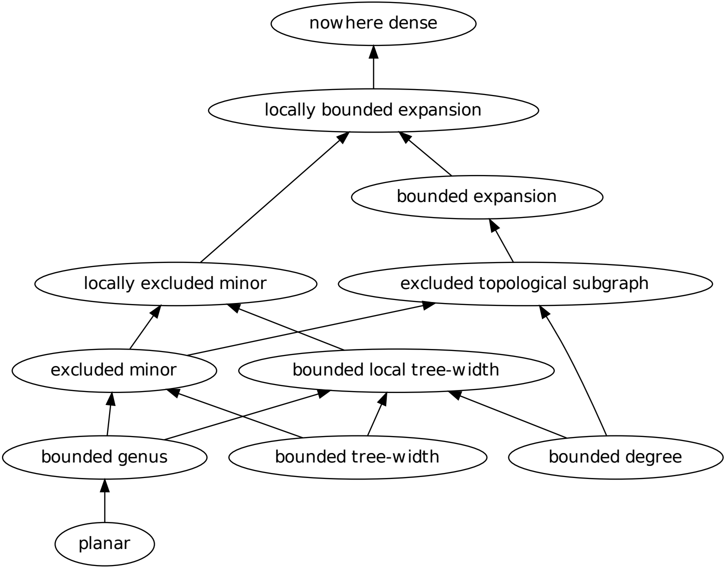
FO Model Checking

- power tool for FO: Gaifman Locality Theorem
  reduces to φ of the form

$$\exists x_1 \dots \exists x_k \left(\bigwedge_i \psi^{(r)}(x_i) \wedge \bigwedge_{i<j} d(x_i, x_j) > 2r \right)$$

  for some $r \in \mathbb{N}$ and r-local formula $\psi^{(r)}$.
- reduces FO-model checking to:
  - evaluating $\psi^{(r)}$ on local parts of the graph and
  - finding $2r$-independent subsets of a set $S \subseteq V$.

FO Model Checking on Monotone Graph Classes



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- basically all known algorithms add the relation in question in a clever way
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Engelmann, Kreutzer, Siebertz; LICS 2012, also Chen, Flum; LICS 2012

Given a graph $G = (V, E)$ of tree-width k, it is possible to compute in fpt a supergraph G' which has a hamiltonian path and has tree-width at most $k + 5$.

- note: $\mathrm{succ} \leftrightarrow$ directed hamiltonian path
- this shows: $<$-inv-MSO model checking on bounded tree-width is fixed-parameter tractable
- works also for bounded clique-width

Model Checking for succ-inv-FO

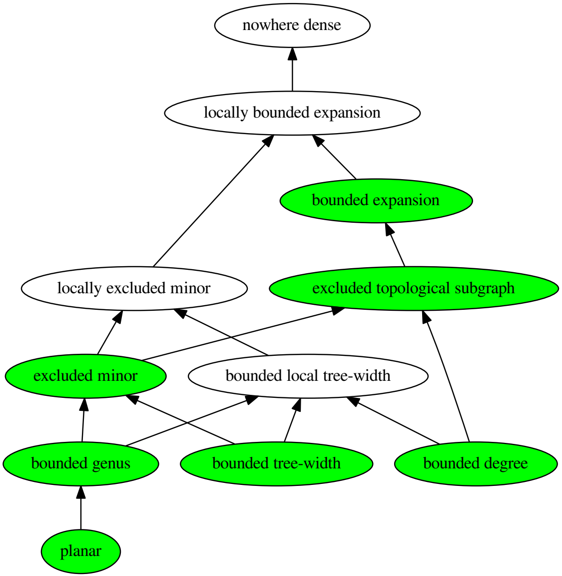
van den Heuvel, Kreutzer, Pilipczuk, Quiroz, Rabinovich, Siebertz, LICS 2017

Model checking for succ-inv-FO is fixed-parameter tractable on classes of graphs of bounded expansion

builds on earlier work:

- planar graphs (Engelmann, Kreutzer, Siebertz; LICS 2012)
- excluded minors (E, Kawarabayashi, Kreutzer; LICS 2013)
- excluded topological minors (E, Kawarabayashi; CSL 2016)

Model Checking for succ-inv-FO



First Step: Reduce to k-walks

E, Kawarabayashi, Kreutzer; LICS 2013

If G contains a k-walk, there is an edge-colouring G' of G such that $\varphi_k(x, y)$ defines a successor-relation on G', for an FO-formula φ_k depending only on k. Moreover G' can be computed from G and the k-walk.

- k-walk: visit each vertex at least once, at most k times
- idea: jump over vertices if they have already been visited
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- somewhat technical, but can define φ_k by induction on k
- k-walk $\leftrightarrow$ spanning tree of degree k'

k-walks: Excluded (Topological) Minors

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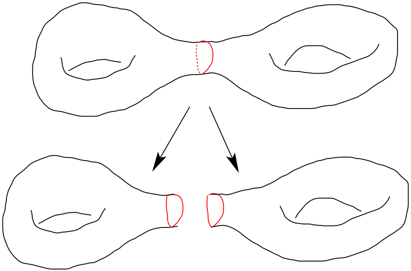

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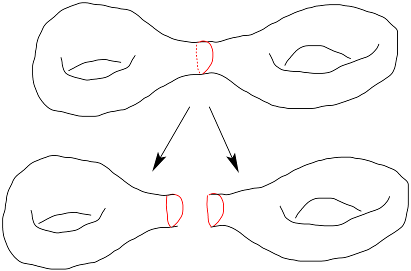

- then lift to excluded (topological) minors using The Big Theorem (Robertson/Seymour, Grohe/Marx)

Graphs of Bounded Expansion

Conclusion

- natural extensions for FO and other logics
- wide gap between classes for which $<$-inv-FO $\equiv$ FO is known and those for which $\prec$ is known
- succinctness?
- model checking for succ-inv-FO on nowhere dense graphs?
- model checking for $<$-inv-FO???