Courcelle's Conjecture, part II: treewidth and cliquewidth

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Based on a joint work with Mikołaj Bojańczyk and Martin Grohe

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Outline

- **First half**:
  - Continuation of Mikołaj's talk:
    Lifting the pathwidth case to the treewidth case.
- **Second half**:
  - Statement of the conjecture for cliquewidth.
  - Highlight of the proof for linear cliquewidth (with MB and MG).

Part I

from pathwidth to treewidth

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- **Final approach**: Reduce the treewidth case to the pathwidth case.
  - **Caveat**: Not a robust approach.

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- **Guidance system**:
  Combinatorial object that provides this functionality.

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A **guidance system** Λ in a graph G is a tuple of rooted forests

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where $V(F_i) = V(G)$ and $E(F_i) \subseteq E(G)$ for each i.

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- For each $u \in V(G)$, define k-tuple $\Lambda(u)$ as

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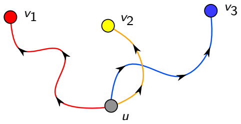
where v_i is the root of the tree of F_i that contains u.



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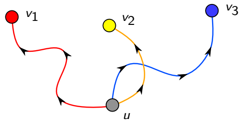

- A vertex subset X is **captured** by Λ if $X \subseteq \Lambda(u)$ for some vertex u.

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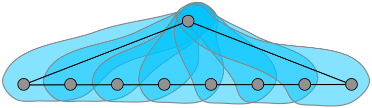

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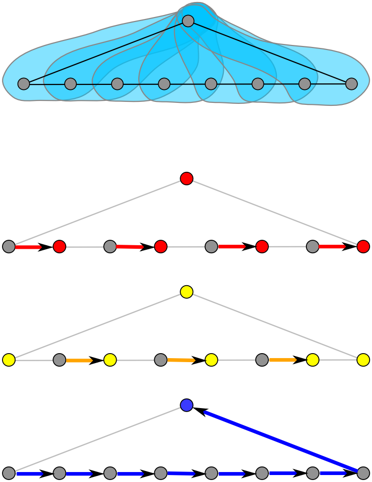

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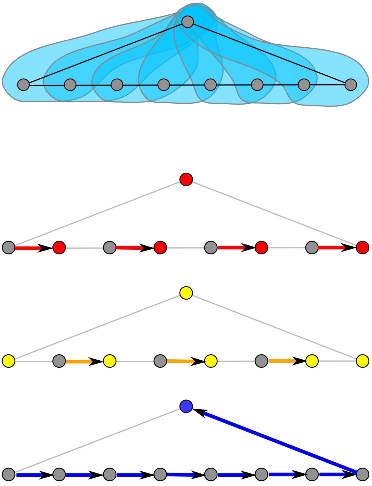

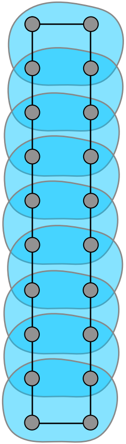

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For every graph G of pathwidth $\leqslant k$, some tree decomposition of G is captured by a guidance system of size $f(k)$.

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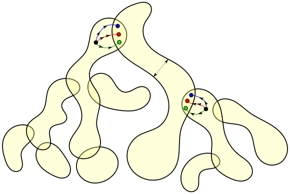
- **Intuition**: Families of subsets captured by small guidance systems can be efficiently guessed in MSO.

Decomposition into low-pathwidth parts

Decomposition into low-pathwidth parts

Every graph G of treewidth k admits a tree decomposition s such that

- the torso of every bag of s has pathwidth bounded by $2k + 1$; and
- the adhesions of s are captured by a guid. system of size $4k^3 + 2k$.



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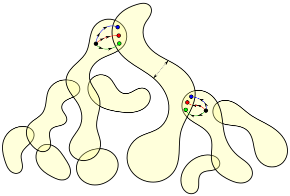

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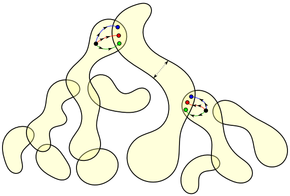

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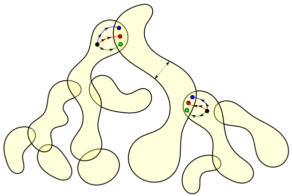

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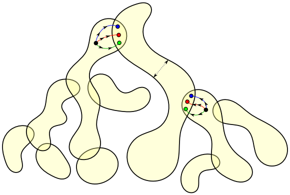

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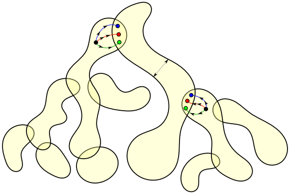
  - Combine all the obtained decompositions along s.



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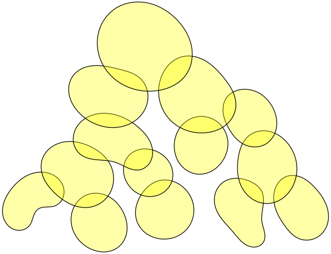

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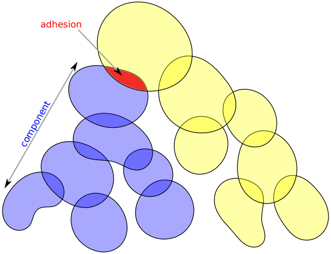
  - **Wlog**: the component at each node is connected its neighborhood is exactly the whole adhesion.



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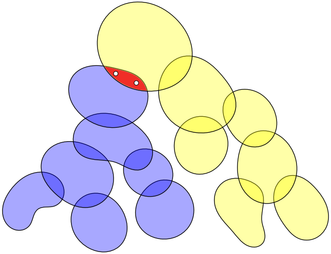

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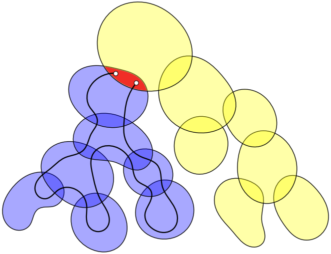

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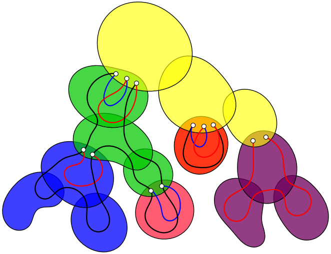

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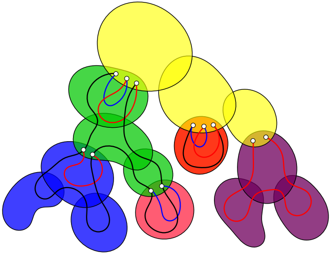

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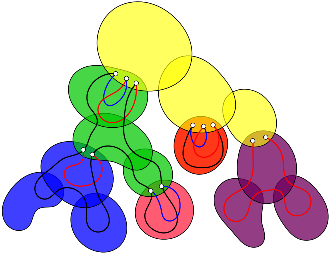

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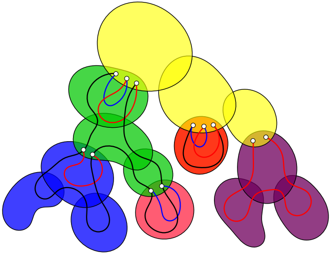
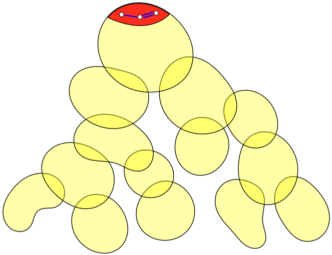
  - We can realize all request in adhesions between pieces using a path system that can be colored with a bounded number of colors.
- **Idea**: Extract pieces by a top-down induction.



Induction

- **Assumptions**:
  - A subtree t of t_0, with top adhesion S.
  - A multiset $\mathcal{R}$ of $\leqslant p(k)$ requests on pairs in S.



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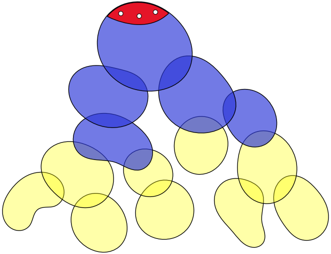

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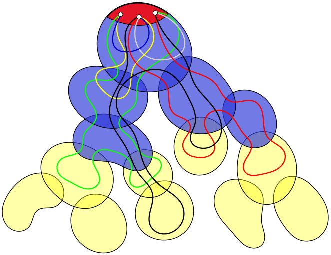

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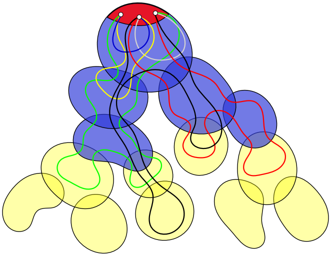
- **Goal achieved** $\Rightarrow$
  Paths can be colored greedily top-down with $p(k) + \binom{k}{2}$ colors.



- **Assumptions**:
  - A subtree t of t_0, with top adhesion S.
  - A multiset $\mathcal{R}$ of $\leqslant p(k)$ requests on pairs in S.
- **Goal**: A prefix X of t such that
  - the torso of the union of bags in X has bnd pathwidth; and
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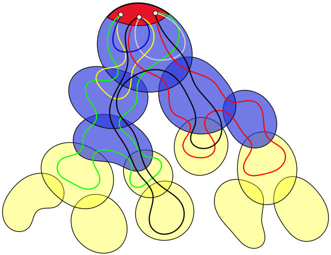

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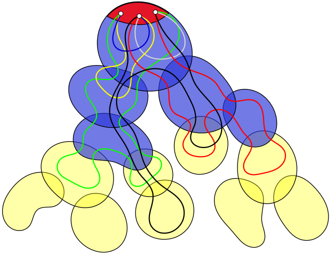
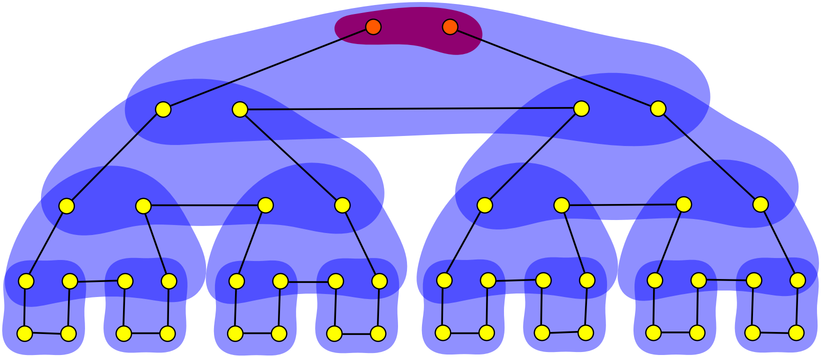
  - **Caveat**: Not quite true, needs a slightly different choice of kings.



Motivating example

- Consider the following tree decomposition of a path, with $p(k)$ requests on the top vertices.



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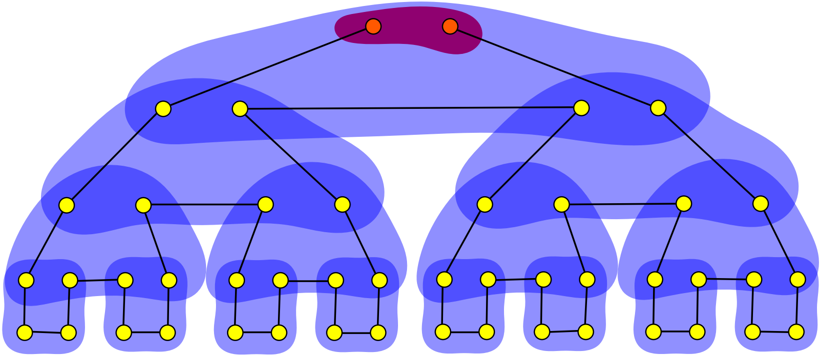

- In this case, X should be the whole t.

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- If achieved, then remaining requests are routed arbitrarily, and

$$p(k) + \binom{k}{2} - \frac{p(k)}{2\binom{k}{2}} < p(k)$$

for a quartic polynomial $p(k)$.

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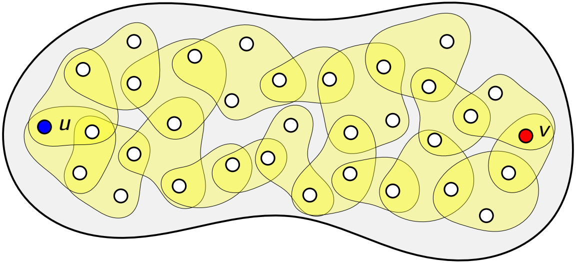

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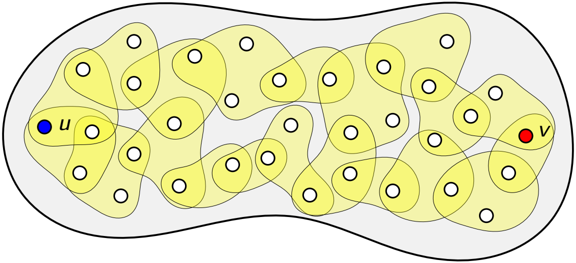

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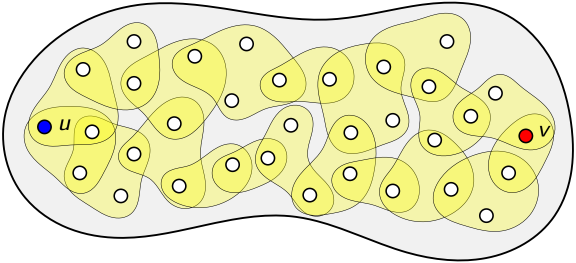

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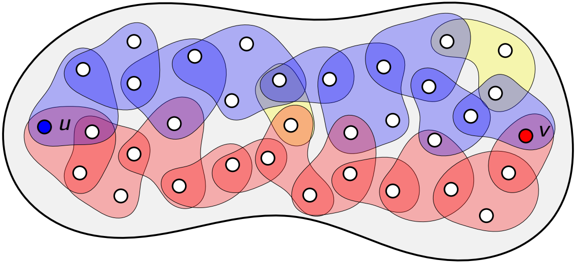

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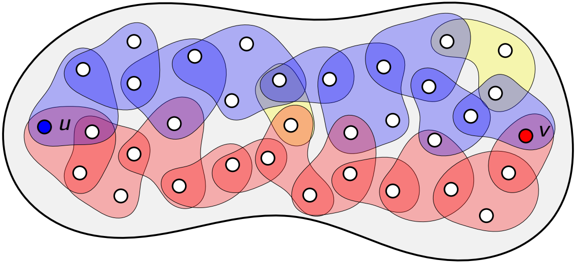

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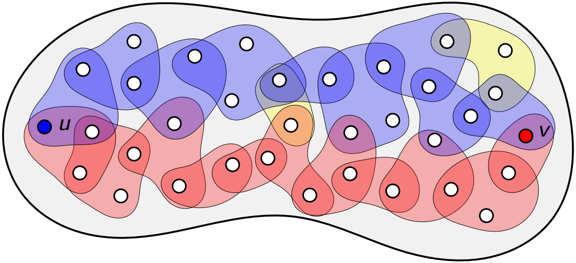
- **Ergo**: If no cutedge, then again $X = \{\mathrm{root}\}$ does the job.



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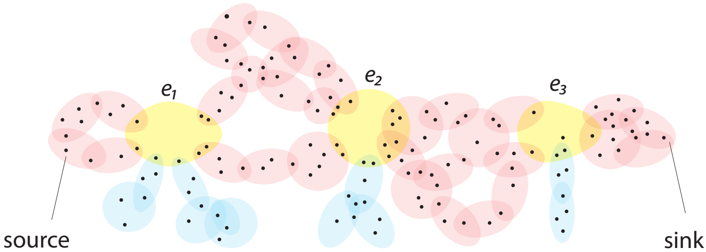
- Otherwise, there is a **sequence** of cutedges.



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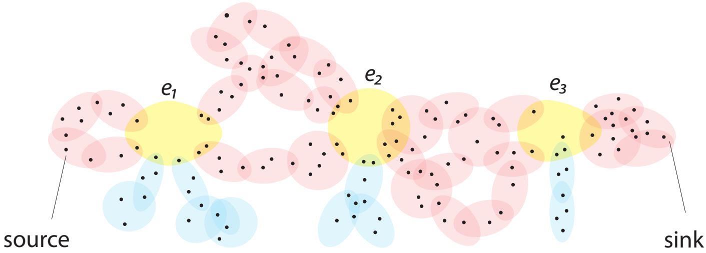

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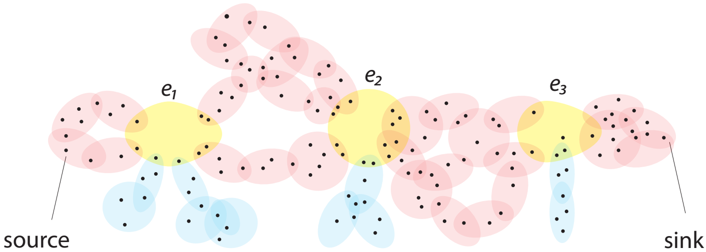

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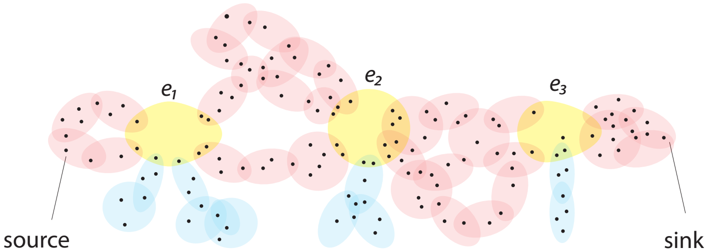

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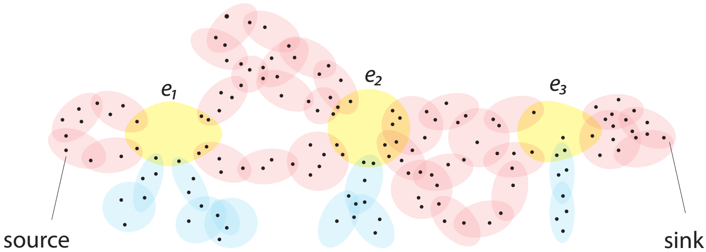

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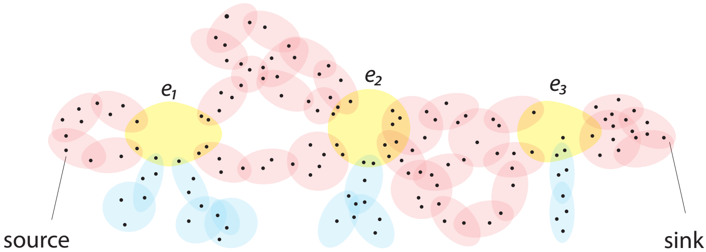
  - This is exactly what happens in the motivating example.



Part II

(linear) cliquewidth

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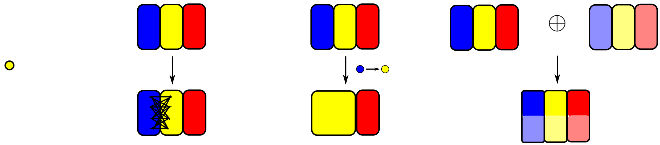

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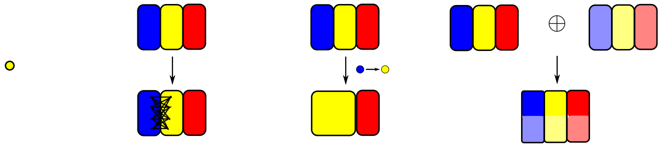

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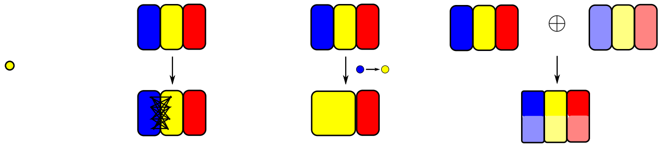

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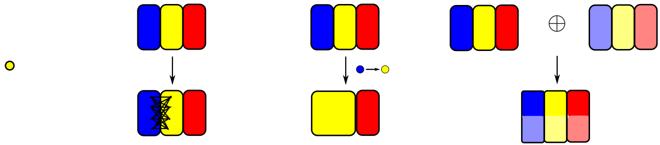

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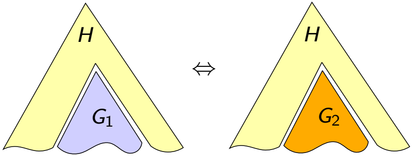

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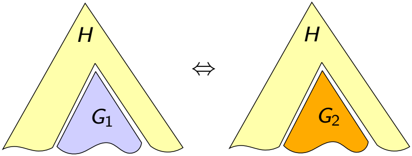

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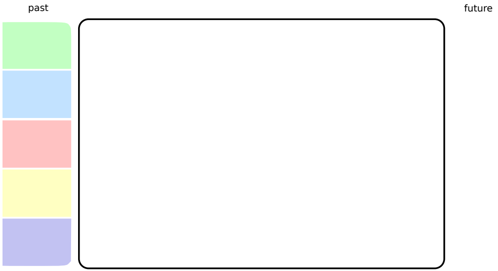

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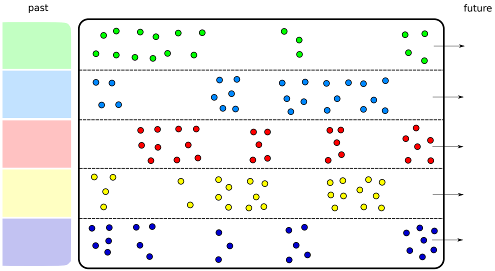

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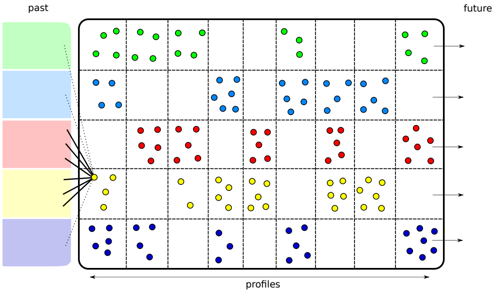

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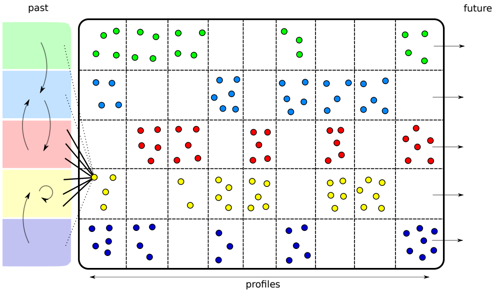

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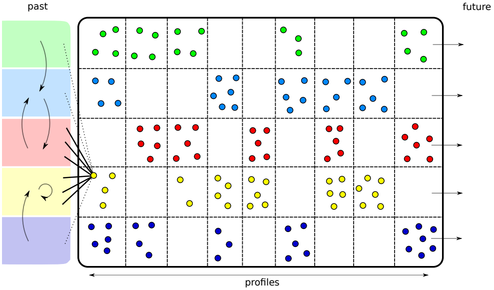

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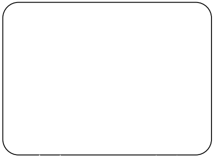

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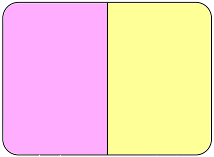

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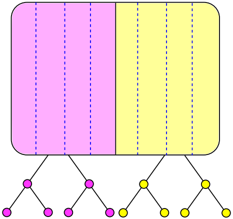

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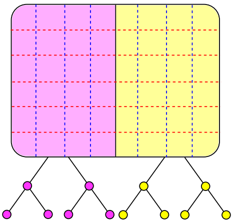

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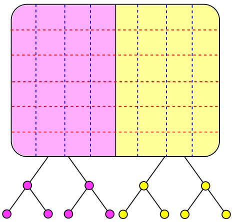

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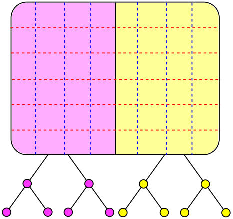

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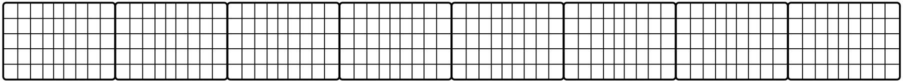

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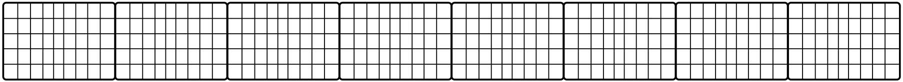

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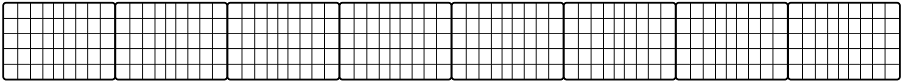

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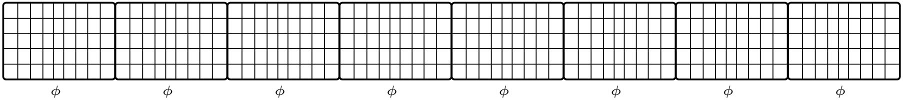

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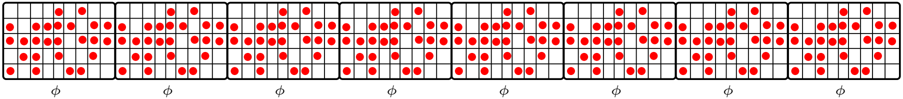

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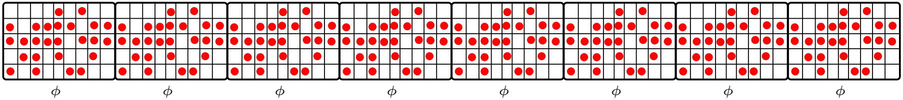

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- **Thank you for your attention!**