The Symmetry Gap in Combinatorial Optimization

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Fixed-Point Logic with Counting

FPC—*Fixed-Point with Counting* is an extension of first-order logic with a *recursion operator* and a mechanism for *counting*.

If $\varphi(x)$ is a formula with free variable x, then $\#x\varphi$ is a term denoting the number of elements satisfying φ.

Formulae of FPC:

- all atomic formulae as in FP;
- $\tau_1 < \tau_2$; $\tau_1 = \tau_2$ where τ_i is a term of numeric sort;
- $\exists x\, \varphi$; $\exists \nu\, \varphi$; where ν is a variable ranging over numbers up to the size of the domain;
- $[\mathbf{lfp}_{X,\mathbf{x},\nu}\varphi](\mathbf{t})$; and
- $\varphi \wedge \psi$; $\neg\varphi$.

Fixed-Point Logic with Counting

FPC is the class of *decision problems* definable in *fixed-point logic with counting*.

> *The decision problems are (isomorphism-closed) classes (or properties) of finite structures (such as graphs, Boolean formulas, systems of equations).*

Every problem in FPC is in P;

Expressive Power of FPC

Most *"obviously"* polynomial-time algorithms can be expressed in FPC.

This includes P-complete problems such as

> CVP—*the Circuit Value Problem*
> *Input: a circuit, i.e. a labelled DAG with source labels from* $\{0, 1\}$*, internal node labels from* $\{\vee, \wedge, \neg\}$*.*
> *Decide: what is the value at the output gate.*

CVP is expressible in FPC.

It is expressible in FPC also for circuits that may include *threshold or counting gates*.

Expressive Power of FPC

Many non-trivial polynomial-time algorithms can be expressed in FPC:

FPC captures all of P over any *proper minor-closed class of graphs* **(Grohe 2010)**

But some cannot be expressed:

- There are polynomial-time decidable properties of graphs that are not definable in FPC. **(Cai, Fürer, Immerman, 1992)**
- *XOR-Sat*, or more generally, solvability of a system of linear equations over a finite field cannot be expressed in FPC. **(Atserias, Bulatov, D. 2009)**

Some NP-complete problems are *provably* not in FPC, including *Sat*, *Hamiltonicity* and *3-colouraiblity*.

Circuit Complexity

A *language* $L \subseteq \{0,1\}^*$ can be described by a family of *Boolean functions*:

$$(f_n)_{n \in \omega} : \{0,1\}^n \to \{0,1\}.$$

Each f_n may be computed by a *circuit* C_n made up of

- Gates labeled by Boolean operators: $\wedge, \vee, \neg$,
- Boolean inputs: $x_1, \ldots, x_n$, and
- A distinguished gate determining the output.

If there is a polynomial $p(n)$ bounding the *size* of C_n, i.e. the number of gates in C_n, the language L is in the class P/poly.

If, in addition, the function $n \mapsto C_n$ is computable in *polynomial time*, L is in P.

Note: For these classes it makes no difference whether the circuits only use $\{\wedge, \vee, \neg\}$ or a richer basis with *threshold* or *majority* gates.

Symmetric Circuits

A Boolean function $f : \{0,1\}^n \to \{0,1\}$ is *symmetric* if it is invariant under *all* permutations of its inputs.

A Boolean function $f : \{0,1\}^{\binom{n}{2}} \to \{0,1\}$ is *graph-invariant* if it is invariant under the natural action of S_n on its inputs.

A circuit C computing $f : \{0,1\}^{\binom{n}{2}} \to \{0,1\}$ is *symmetric* if the action of any permutation in S_n on the inputs can be extended to an *automorphism* of C.

A graph property is in FPC *if, and only if,* it is decided by a P-uniform family of *symmetric* circuits using *symmetric gates*.

Weisfeiler-Leman Equivalences

The k-*dimensional Weisfeiler-Leman* equivalence relation is an *overapproximation* of the isomorphism relation.

If G, H are n-vertex graphs and $k < n$, we have:

$$G \cong H \quad \Leftrightarrow \quad G \equiv^n H \quad \Rightarrow \quad G \equiv^{k+1} H \quad \Rightarrow \quad G \equiv^k H.$$

$G \equiv^k H$ is decidable in time $n^{O(k)}$.

It has many equivalent characterisations arising from

- *combinatorics* **(Babai)**
- *logic* **(Immerman-Lander)**
- *algebra* **(Weisfeiler; Holm)**
- *linear optimization* **(Atserias-Maneva; Malkin)**

Weisfeiler-Leman Equivalences

$G \equiv^k H$ iff G and H cannot be distinguished by a sentence of first-order logic with *counting quantifiers* using only $k+1$ variables.

G and H are not distinguished by the coarsest partition of the k-tuples of G into classes $P_1, \ldots, P_t$ satisfying:

> *two tuples* $\mathbf{u}$ *and* $\mathbf{v}$ *in the same class* P_i *cannot be distinguished by counting the number of substitutions we can make in them to get a tuple in class* P_j*.*

Graph Isomorphism Integer Program

Yet another way of approximating the *graph isomorphism relation* is obtained by considering it as a *0/1 linear program*.

If A and B are adjacency matrices of graphs G and H, then $G \cong H$ if, and only if, there is a *permutation matrix* P such that:

$$PAP^{-1} = B \quad \text{or, equivalently} \quad PA = BP$$

Introducing a variable x_{ij} for each entry of P and adding the constraints:

$$\sum_i x_{ij} = 1 \quad \text{and} \quad \sum_j x_{ij} = 1$$

we get a system of equations that has a *0-1 solution* if, and only if, G and H are isomorphic.

Fractional Isomorphism

To the system of equations:

$$PA = BP; \quad \sum_i x_{ij} = 1 \quad \text{and} \quad \sum_j x_{ij} = 1$$

add the inequalities

$$0 \leq x_{ij} \leq 1.$$

Say that G and H are *fractionally isomorphic* ($G \cong^f H$) if the resulting system has *any real solution*.

$G \cong^f H$ if, and only if, $G \equiv^1 H$.

(Ramana, Scheiermann, Ullman 1994)

Sherali-Adams Hierarchy

If we have any *linear program* for which we seek a *0-1 solution*, we can relax the constraint and admit *fractional solutions*.

The resulting linear program can be solved in *polynomial time*, but admits solutions which are not solutions to the original problem.

Sherali and Adams (1990) define a way of *tightening* the linear program by adding a number of *lift and project* constraints.

Sherali-Adams Hierarchy

The kth *lift-and-project* of a linear program is defined as follows:

For each constraint $\mathbf{a}^T\mathbf{x} \leq b$ in the linear program, and each set I of variables with $|I| < k$ and $J \subseteq I$, multiply the constraint by

$$\prod_{i \in I \setminus J} x_i \prod_{j \in J} (1 - x_j)$$

and then *linearize* by replacing x_i^2 by x_i and $\prod_{j \in K} x_j$ by a new variable y_K for each set K (along with constraints: $y_\emptyset = 1$, $y_{\{x\}} = x$ and $y_K \leq y_{K'}$ for $K' \subseteq K$).

Say that $G \cong^{f,k} H$ if the kth lift-and-project of the *isomorphism program* on G and H admits a solution.

Sherali-Adams Isomorphism

For each k

$$G \equiv^k H \quad \Rightarrow \quad G \cong^{f,k} H \quad \Rightarrow \quad G \equiv^{k-1} H$$

(Atserias, Maneva 2012)

For $k > 2$, the reverse implications fail. **(Grohe, Otto 2012)**

Counting Width

For any class of structures $\mathcal{C}$, we define its *counting width* $\nu_{\mathcal{C}} : \mathbb{N} \to \mathbb{N}$ so that

> *$\nu_{\mathcal{C}}(n)$ is the least k such that $\mathcal{C}$ restricted to structures with at most n elements is closed under $\equiv^k$.*

Every class in FPC has counting width bounded by a *constant*.

3-Sat, *XOR-Sat*, *3-Colourability* all have counting width $\Omega(n)$.

FPC-Reductions

If $\mathcal{C} \leq_{\mathsf{FPC}} \mathcal{D}$ then

$$\nu_{\mathcal{D}} = \Omega(\nu_{\mathcal{C}}^{1/d}).$$

If the reduction takes $\mathcal{C}$-instances to $\mathcal{D}$-instances of *linear size*, then

$$\nu_{\mathcal{D}} = \Omega(\nu_{\mathcal{C}}).$$

Known linear lower bounds follow from $\nu_{\mathsf{XOR\text{-}Sat}} = \Omega(n)$.

Linear Programming

Linear Programming is an important algorithmic tool for solving a large variety of optimization problems.

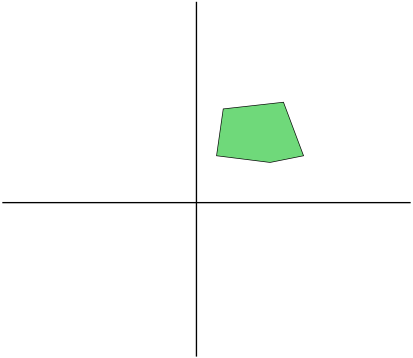
It was shown by **(Khachiyan 1980)** that linear programming problems can be solved in polynomial time.
We have a set C of *constraints* over a set V of *variables*.
Each $c \in C$ consists of $a_c \in \mathbb{Q}^V$ and $b_c \in \mathbb{Q}$.

Feasibility Problem: Given a linear programming instance, determine if there is an $x \in \mathbb{Q}^V$ such that:

$$a_c^T x \leq b_c \quad \text{for all } c \in C$$

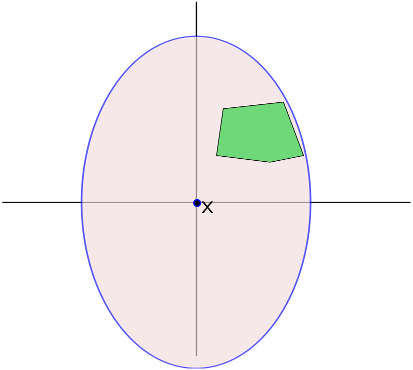
We show that this, and the corresponding *optimization problem* are expressible in FPC.

Ellipsoid Method



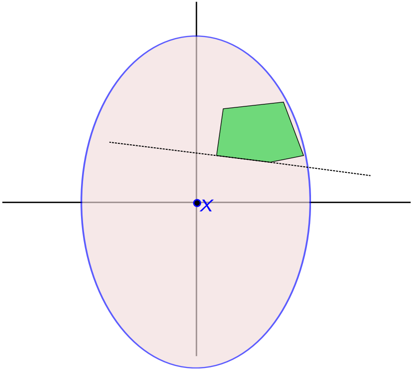
The set of constraints determines a *polytope*

Ellipsoid Method



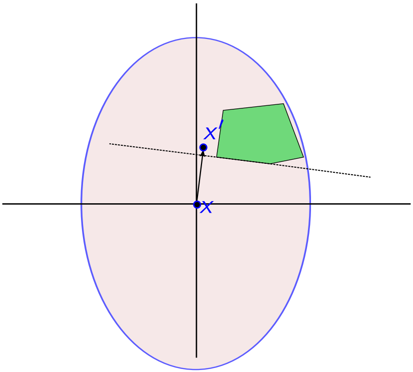
Start at the origin and calculate an *ellipsoid* enclosing it.

Ellipsoid Method



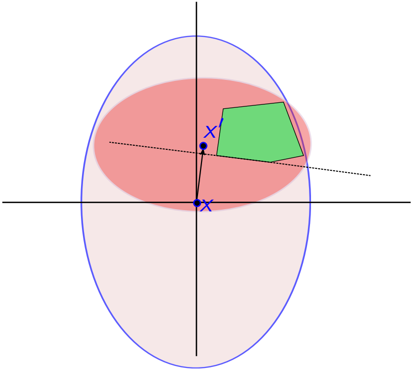
If the centre is not in the polytope, choose a constraint it *violates*.

Ellipsoid Method



Calculate a new *centre*.

Ellipsoid Method



And a new ellipsoid around the centre of at most *half* the volume.

Ellipsoid Method in FPC

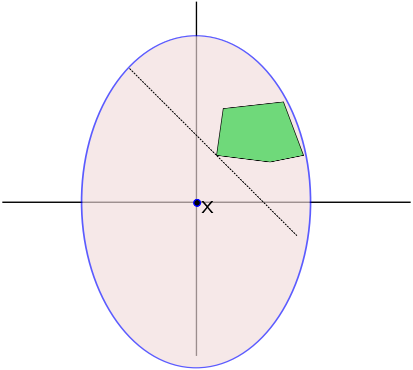
We can encode all the calculations involved in FPC.

This relies on expressing algebraic manipulations of *unordered* matrices.

What is not obvious is how to *choose* the violated constraint on which to project.

However, the ellipsoid method works as long as we can find, at each step, some *separating hyperplane*.

Ellipsoid Method in FPC



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So, we can take:

$$(\sum_{c \in S} a_c)^T x \leq \sum_{c \in S} b_c$$

where S is the *set* of all violated constraints.

Separation Oracle

More generally, the ellipsoid method can be used, even when the *constraint matrix* is not given explicitly, as long as we can always determine a *separating hyperplane*.

In particular, the polytope represented may have *exponentially many* facets.

We show that as long as the *separation oracle* can be defined in FPC, the corresponding *optimization problem* can be solved in FPC.

Graph Matching

Recall, in a *graph* $G = (V, E)$ a matching $M \subset E$ is a set of edges such that each vertex is incident on *at most* one edge in M.

(Blass, Gurevich, Shelah 1999) showed that for *bipartite* graphs this is definable in FPC.

We consider the more general problem of determining the *maximum weight* of a matching in a *weighted graph*:

$$G = (V, E) \quad w : E \to \mathbb{Q}_{\geq 0}$$

The Matching Polytope

(Edmonds 1965) showed that the problem of finding a *maximum weight matching* in $G = (V, E)$ $\quad w : \mathbb{Q}^E_{\geq 0}$ can be expressed as the following linear programming problem

$$\max w^\top y \qquad \text{subject to}$$

$$Ay \leq 1^V,$$

$$y_e \geq 0, \;\; \forall e \in E,$$

$$\sum_{e \in E \cap W^2} y_e \leq \frac{1}{2}(|W| - 1), \;\; \forall W \subseteq V \text{ with } |W| \text{ odd},$$

Matching in FPC

We show that a *separation oracle* for this polytope is definable by an FPC formula interpreted in the weighted graph G.

As a consequence, there is an FPC formula defining the *size* of the maximum matching in G.

Note that this does not allow us to define an *actual* matching.

Finite Valued CSPs

Finite Valued CSPs generalize various Max-CSP problems.

Such a problem is given by a finite domain D and a collection Γ of functions $f : D^k \to \mathbb{Q}_+$.

An instance is a set V of variables along with constraints $c = (\mathbf{x}, f, w)$ for $\mathbf{x} \in V^k$ and $w \in \mathbb{Q}_+$.

Find an assignment $h : V \to D$ which minimizes

$$\sum_c wf(h\mathbf{x})$$

As usual, we get a decision problem by including a threshold t.

Linear Programming Relaxations

Each instance I of (D, Γ) can be turned into a linear program $\mathrm{BLP}(I)$:
Set of variables V, domain D, constraints $c = (x, R)$

$$\max \sum_{c \in C} \sum_{d \in R^{\Gamma}} \lambda_{c,d} \quad \text{where } c = (x, R), \text{ s.t.}$$

$$\sum_{d \in D^{|x|}; d_i = a} \lambda_{c,d} = \mu_{x_i, a} \qquad \forall c \in C, a \in D, i \in [|x|]$$

$$\sum_{a \in D} \mu_{v,a} = 1 \qquad \forall v \in V$$

Lift and Project Hierarchies

Given a *polytope* $\mathcal{K}$ for *integer* optimization problem, we can get a better approximation of the *convex hull* of the integer points by means of *lift-and-project* programs.

The general idea is to add new variables $y_{x_1,\ldots,x_t}$ to denote the product $x_1 \cdots x_t$ and add linear (or semi-definite) constraints to try and force this meaning.

We get hierarchies as t increases:

- *Sherali-Adams*: $\mathsf{SA}_t(\mathcal{K})$
- *Lovasz-Schrijver*: $\mathsf{LS}_t(\mathcal{K})$
- *Lasserre*: $\mathsf{Las}_t(\mathcal{K})$

Of these, the last is the strongest.

Dichotomy

For each Γ and t, there is an FPC interpretation that takes an instance I of $\mathrm{CSP}(\Gamma)$ to the tth level of the Lasserre hierarchy over $\mathrm{BLP}(I)$.

For every finite-valued CSP (D, Γ), the counting width of the corresponding decision problem is either $O(1)$ or $\Omega(n)$.

In the former case the problem can be solved exactly by its *linear programming relaxation*.

Every finite-valued CSP (D, Γ) that is *not* solved by its basic linear programming relaxation is not solved exactly by $o(n)$-level Lasserre lift-and-project.

Symmetry Gap in Constraint Optimization

Gaussian elimination is *not* a symmetric algorithm.

And, it cannot be made symmetric—this is established by the *XOR-Sat* lower bounds.

The ellipsoid method can be made symmetric.

Lower bounds on combinatorial optimization methods can be derived from this gap.

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