Online Algorithms for Multi-commodity Network Design

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Thanks Alina for sharing slides!

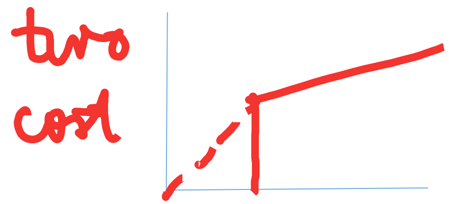
Online Network Design: a quick review

- (Un)directed graph **G = (V, E)** given offline

- Every subgraph has an associated cost
(fixed edge cost, fixed node cost, buy at bulk, two cost)

- Terminals/terminal pairs arrive online

- Algorithm has to connect:
  - Terminal to a root (single sink)
  - Terminal pair (multi-commodity)

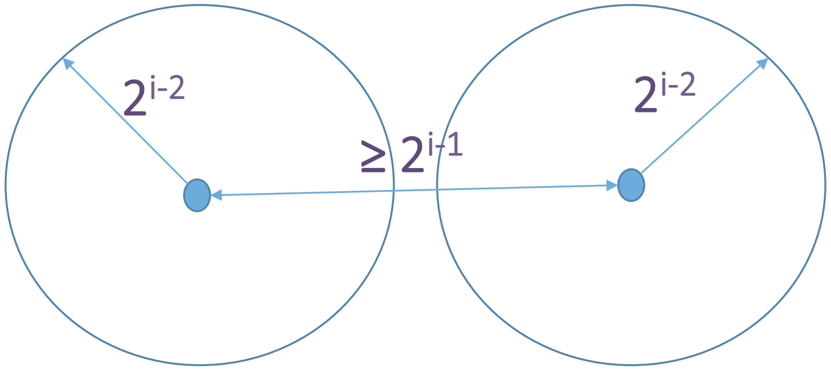


- Competitive Ratio = Online Alg / (Offline) Opt

Online Network Design: a bit of history

- Edge-weighted Steiner tree (Imase-Waxman, '91)

- Greedy algorithm
- Charge cost of algorithm to **log n** layers of duals
  - Layer **i** has ball of radius $\mathbf{2^{i-2}}$ centered at **v** if cost of connecting **v** is (approximately) $\mathbf{[2^{i-1}, 2^{i}]}$

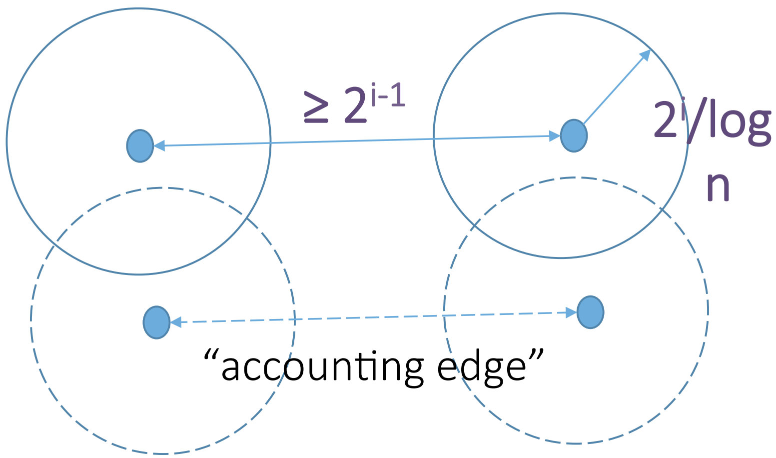


Competitive ratio:
O(log n)

Online Network Design: a bit of history

- Edge-weighted Steiner forest (Awerbuch-Azar-Bartal, ’94)

- Greedy algorithm
- Charge cost of algorithm to **log n** layers of duals
  - Layer **i** has ball of radius $\mathbf{2^i/\log n}$ centered at $\mathbf{s_i}$ or $\mathbf{t_i}$ if cost of connecting this pair is (approximately) $\mathbf{[2^{i-1}, 2^i]}$



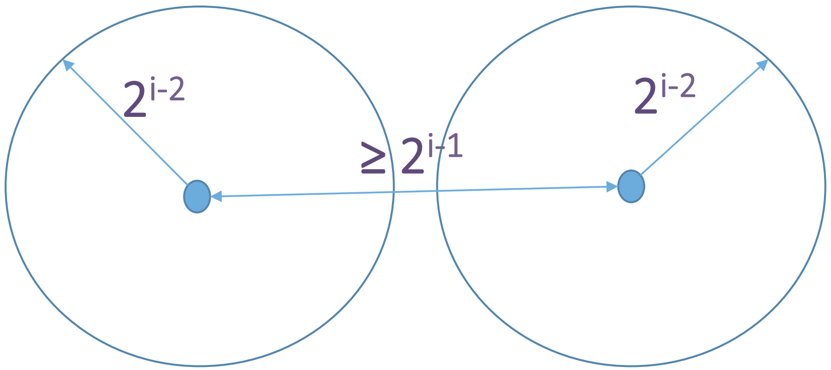
Competitive ratio:
$O(\log^2 n)$

Modified greedy
[Berman-Coulston, ‘97]:
O(log n)

Online Network Design: more recent history

- Node-weighted Steiner tree (Naor-P.-Singh, '11)

- Greedy algorithm fails …
  - … because node-weighted Steiner tree captures set cover
- Main idea: minimal cost sharing
  - connecting path = greedy path + one shared vertex
  - use online set cover for shared vertex, greedy for the rest

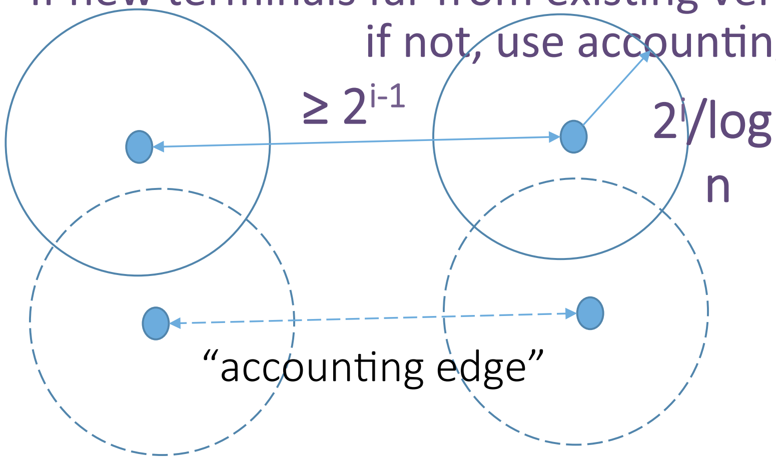


Competitive ratio: O(poly log n)

Online Network Design: more recent history

- Node-weighted Steiner forest (Hajiaghayi-Liaghat-P., '13)

- Node-weighted Steiner tree approach fails …
  - … sharing on **log n** vertices required: label cover hard
- Charge cost of algorithm to **log n** layers of ~~duals~~ set cover instances
  - Optimum of each set cover instance lower bounds the primal
  - If new terminals far from existing vertices, optimum set cover cost changes; if not, use accounting edge as earlier



Competitive ratio:
O(poly log n)

Online Network Design: moral of the story

- Multi-commodity problems have historically required new techniques
- Not any more ☺

Single-sink algorithm

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This talk

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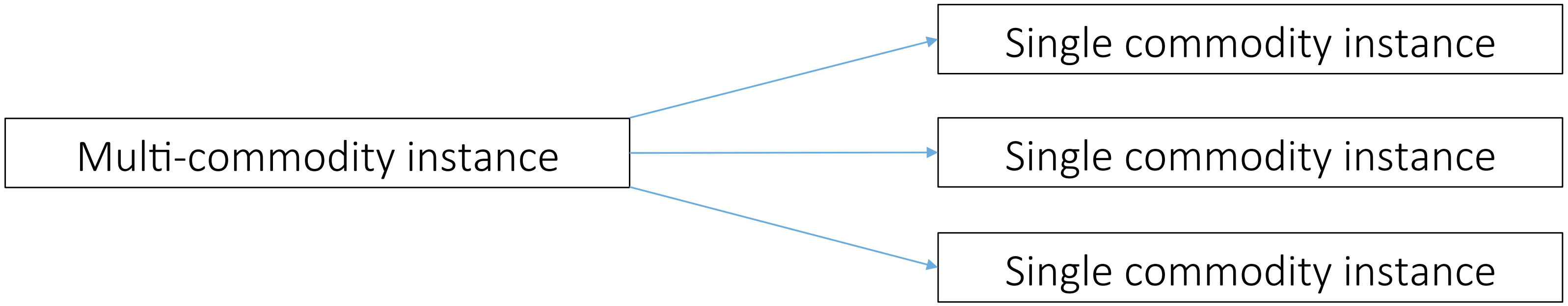
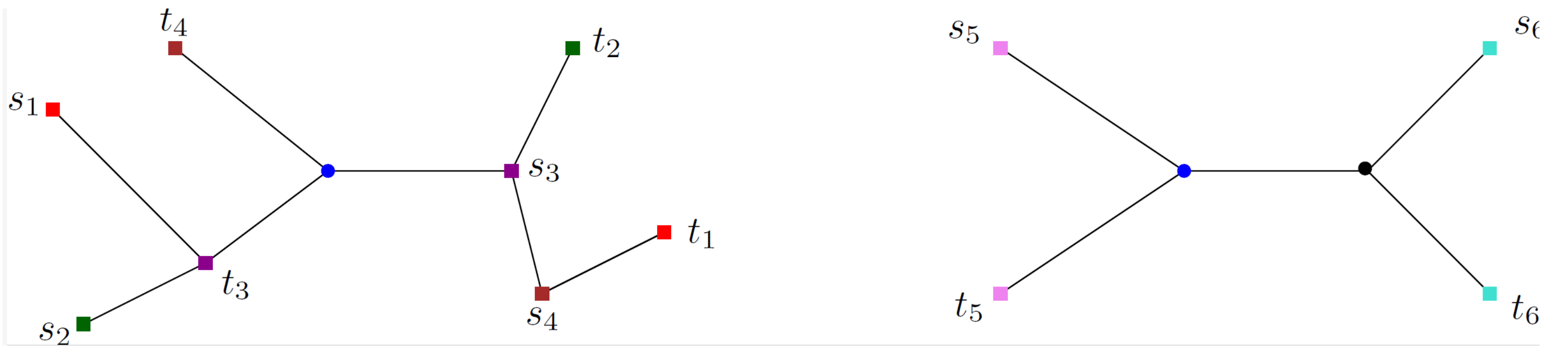
Multi-commodity algorithm

What about offline?

- Junction Trees [Chekuri, Kortsarz, Hajiaghayi, Salavatipour ’06]

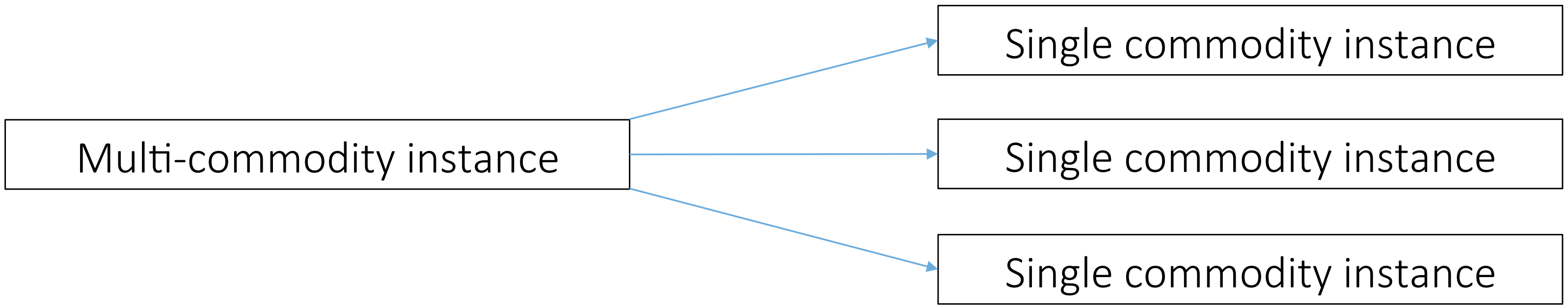


  - Route a subset of the pairs via a junction vertex; overall cost is sum of individual junction tree costs



What about offline?

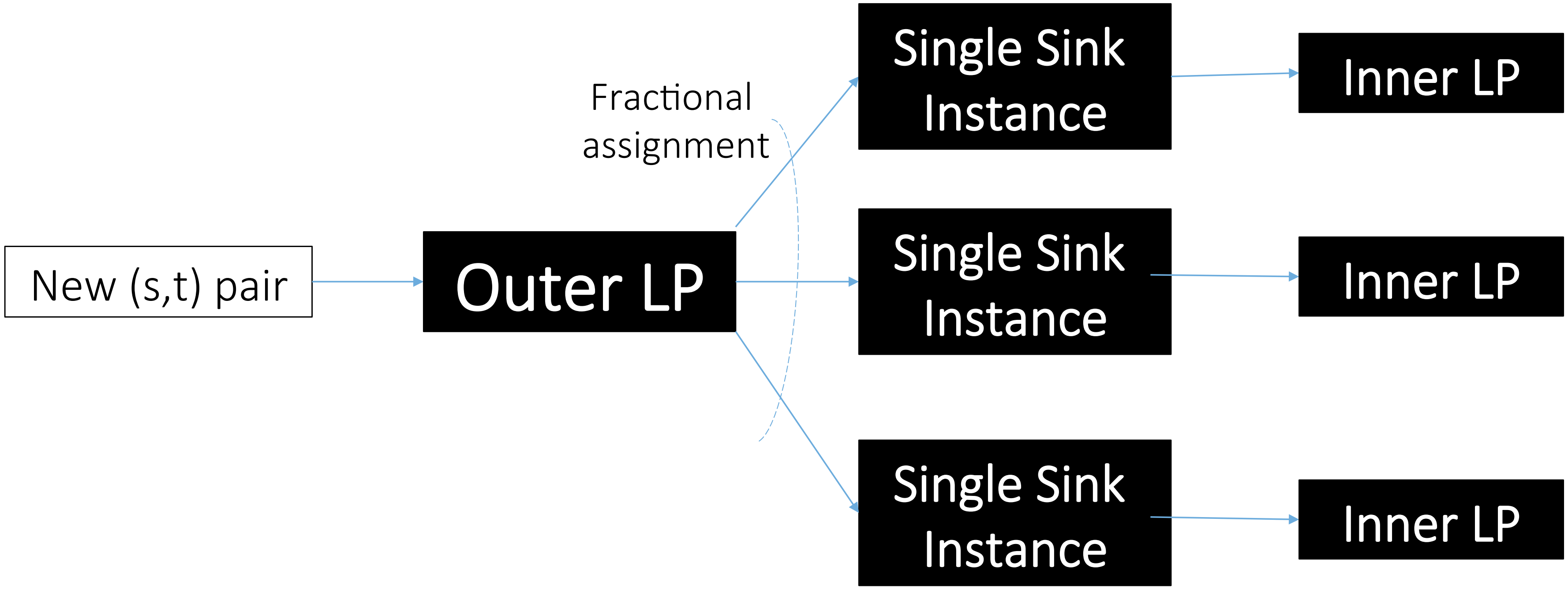
- Junction Trees [Chekuri, Kortsarz, Hajiaghayi, Salavatipour ’06]



  - Route a subset of the pairs via a junction vertex; overall cost is sum of individual junction tree costs
  - Existence of a near-optimal junction tree solution
    - easy for Steiner forest, more involved for buy at bulk cost functions
  - Partition the terminal pairs via a greedy covering algorithm, where each greedy step runs a min-density single sink algorithm

Online junction trees

- Arguably even more useful! The **log** set cover loss is not limiting
- Focus on obtaining a fractional solution first



The Composite (Outer + Inner) LP for buy at bulk

two cost

$$\text{minimize} \sum_{r\in V}\sum_{e\in E} c_e x_e^r + \sum_{(s_i,t_i)\in\mathcal{X}}\sum_{r\in R}\sum_{e\in E} \ell_e \left(f^r_{(e,s_i)} + f^r_{(e,t_i)}\right)$$

Outer LP

$$\text{s.t} \sum_{r\in V} z_{ir} \geq 1 \qquad \forall i$$

$$z_{ir} \geq 0$$

assignment variables

inner LPs

$\{f^r_{(e,s_i)}\}$ define a flow from s_i to r of value z_{ir} in G^{up}	$\forall i, \forall r \in V$
$\{f^r_{(e,t_i)}\}$ define a flow from r to t_i of value z_{ir} in G^{down}	$\forall i, \forall r \in V$
$f^r_{(e,s_i)} \leq x^r_e$	$\forall i, \forall e, \forall r \in V$
$f^r_{(e,t_i)} \leq x^r_e$	$\forall i, \forall e, \forall r \in V$
$x^r_e \geq 0,\ f^r_{(\cdot)} \geq 0$	

Solve the Composite LP

- Use the online primal-dual framework of [Buchbinder, Naor, and others]
- Some new ideas, does not fall in previous packing/covering/mixed templates
  - Outer LP runs multiplicative weight updates, as usual
  - Flow-type constraints in inner LP, uses a min cost (max) flow gadget

Rounding Pains

- Outer LP: easy, randomized rounding
- Inner LP: no idea! (how to round Steiner tree LP online?)
- Do not round! ☺
  - Use bounded integrality gap (from offline rounding known) to claim that each inner LP has integer solution of ``small value''
  - Run a separate online Steiner tree instance for each inner LP in parallel

Results

- First non-trivial online algorithms for multi-commodity buy at bulk network design with edge costs, node costs, and in directed graphs

- Simpler alternative proof for online Steiner forest

- Easily extends to other cost frameworks such as prize-collecting problems

Open Problems

- Node-weighted algorithms are quasi-polynomial time using this template, but polynomial time using specific algorithms
- Can we avoid losing a **log D** factor for non-uniform demands, where **D** is max-to-min demand ratio?
- How do you round (even) the Steiner tree LP online?

Thank You

Questions?