Inventory Problems with Submodular or Routing Costs

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Joint work with Cong Shi

Inventory Optimization

- Demands over time horizon T

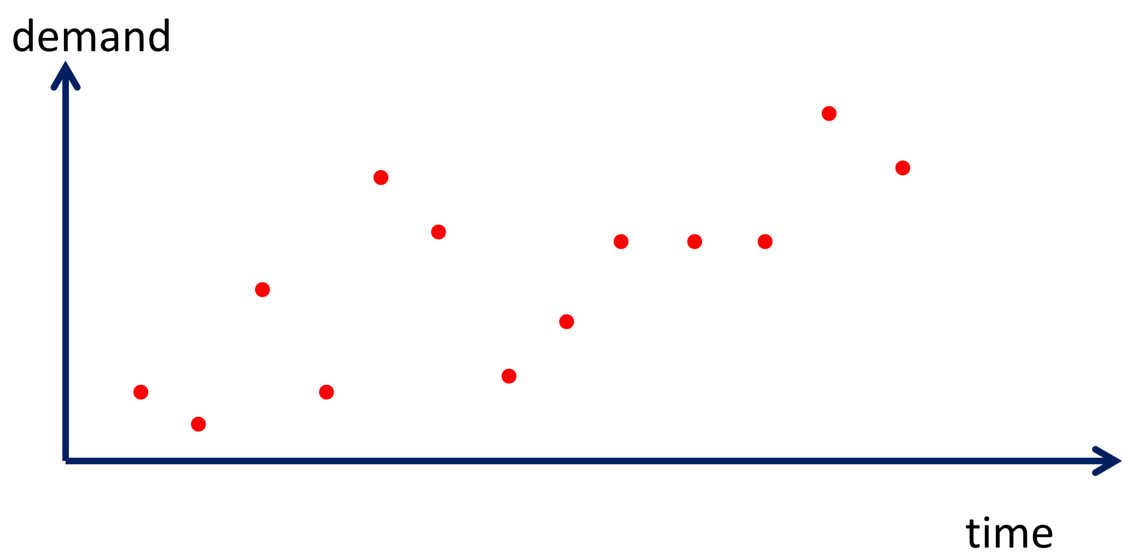
  uniform in each period

  non-uniform ←

- Holding costs

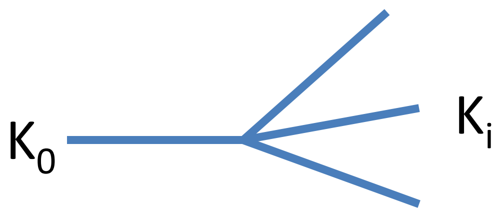
  linear ←

  general



Joint Replenishment Problem

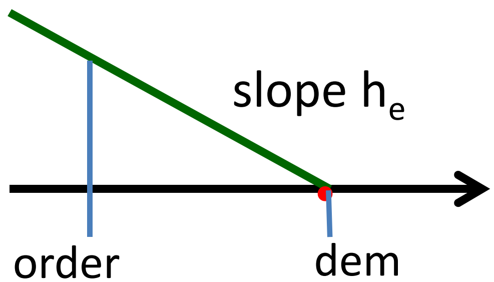
- n items with demands over horizon T
- Ordering cost "additive joint"



Order items S $\Rightarrow$ $K_0 + \sum_{i \in S} K_i$

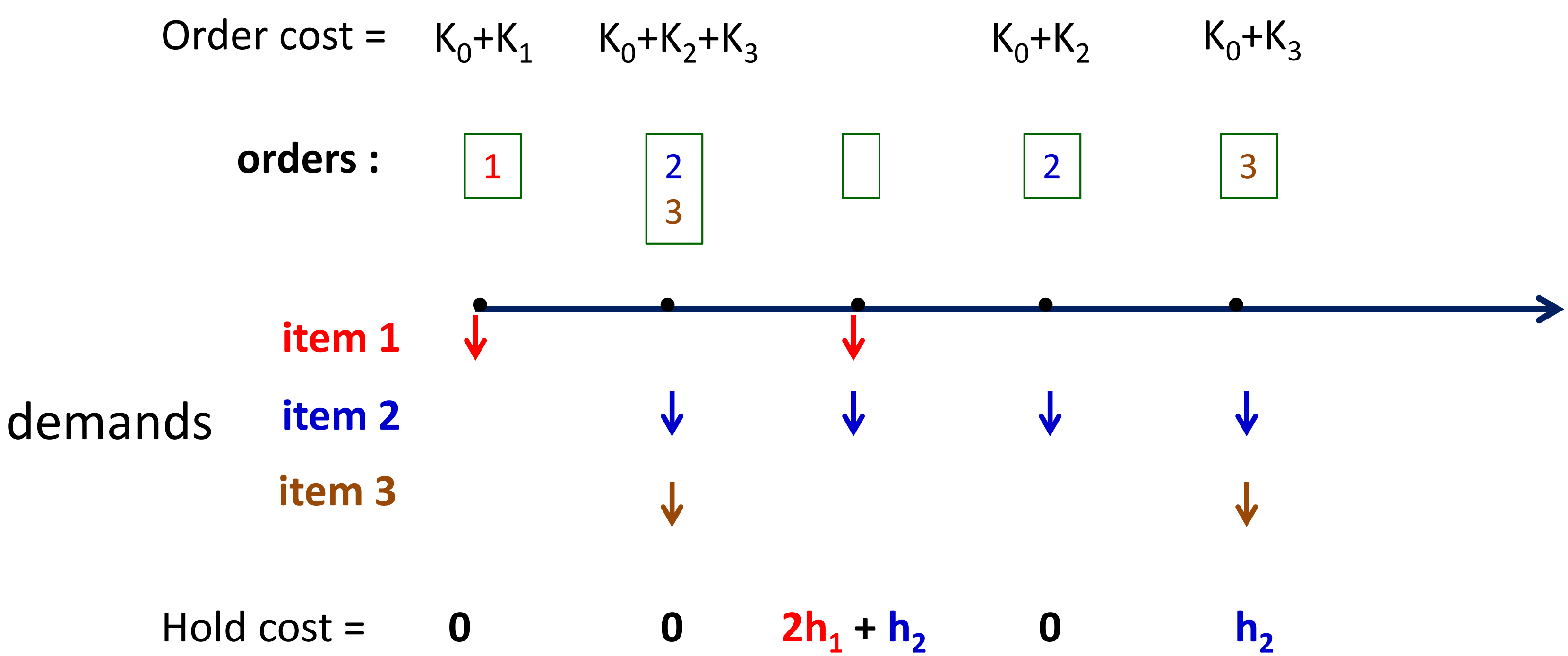
Unlimited quantity

- Holding cost



Minimize ordering + holding

JRP Example


Order cost = K_0+K_1 $K_0+K_2+K_3$ K_0+K_2 K_0+K_3
orders :
1
2
3
2
3
demands
item 1
item 2
item 3
Hold cost = **0** **0** $\mathbf{2h_1 + h_2}$ **0** $\mathbf{h_2}$

JRP Known Results

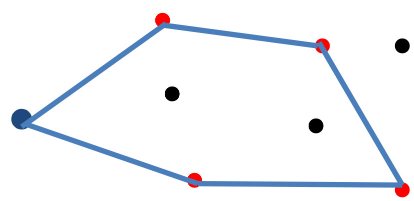
- NP-hard [Arkin Joneja Roundy '89]
- 1.02-approx for uniform demand [Roundy '85]
- 1.79 approximation algorithm for non-uniform
  [Bienkowski Byrka Chrobak Jez Sgall '13]
  [Levi Roundy Shmoys Sviridenko '08]

Submodular JRP

- Order S $\Rightarrow$ cost f(S) for submodular f : $2^{[n]} \rightarrow R$
- 1.02-approx for uniform demand [Federgruen Zheng '92]
- O(log nT) approximation for non-uniform
  [Cheung Elmachtoub Levi Shmoys '15]
- O(1) approximation for special f
  cardinality, tree, laminar

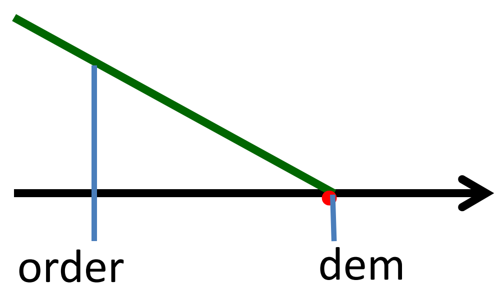
Inventory Routing Problem

- n demand locations in metric, depot r
- Demands over horizon T
- Routing cost: visit locations by TSP



Visit S $\Rightarrow$ TSP(S) cost
Unlimited quantity

- Holding cost



Minimize routing + holding costs

IRP Known Results

- Lots of computational work

  Eg. [Coelho Cordeau Laporte '14]

- O(1) approximation when restricted to "nested periodic" policies [Fukunaga Nikzad Ravi '14]

- O(log n) approximation by tree embedding

  JRP itself 2-level tree

Our Result

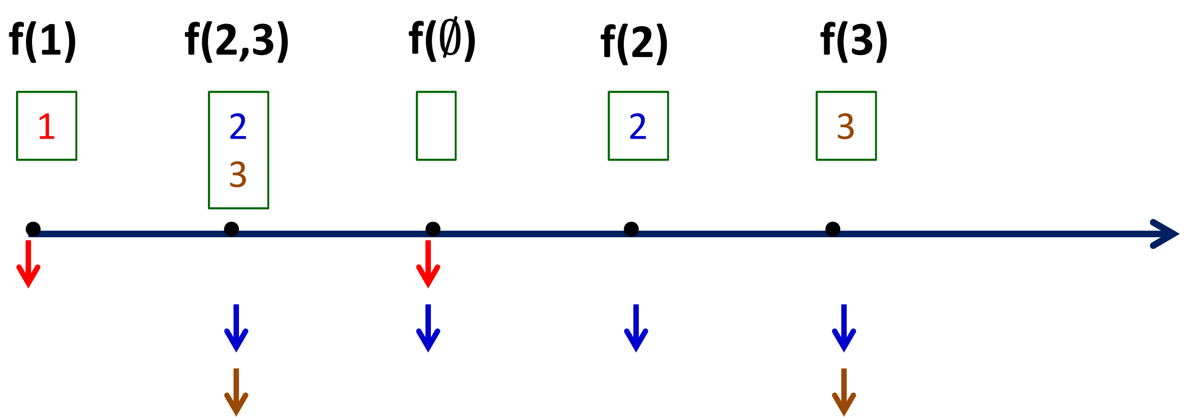
$O\left(\frac{\log T}{\log \log T}\right)$ approximation algorithm

- For both submodular JRP and IRP
- Also for “polynomial” holding costs

Unified Inventory Problem

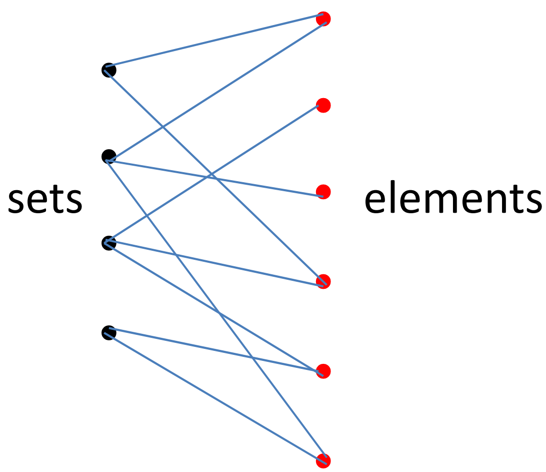
- N elements
- T time periods
- Demand d_{er} units for e at r
- **Ordering cost f : $2^N \rightarrow$ R**
- Holding cost h_e / unit / time

Min ordering + holding costs



Uniform Ordering Cost

- If ordering cost *time-varying* then set-cover hard



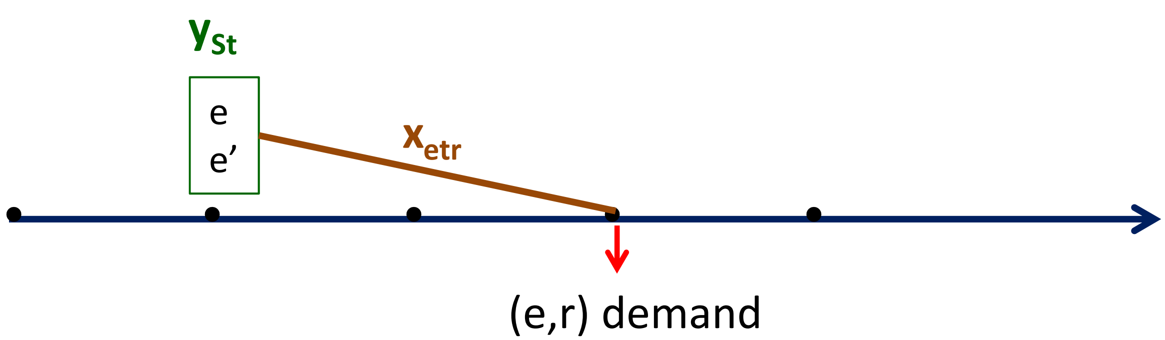
- Need uniform f for sub-logarithmic ratio

LP relaxation

y_{St} = order exactly S at time t

x_{etr} = satisfy demand (e,r) by time t order

Similar to [Cheung et al. '15]



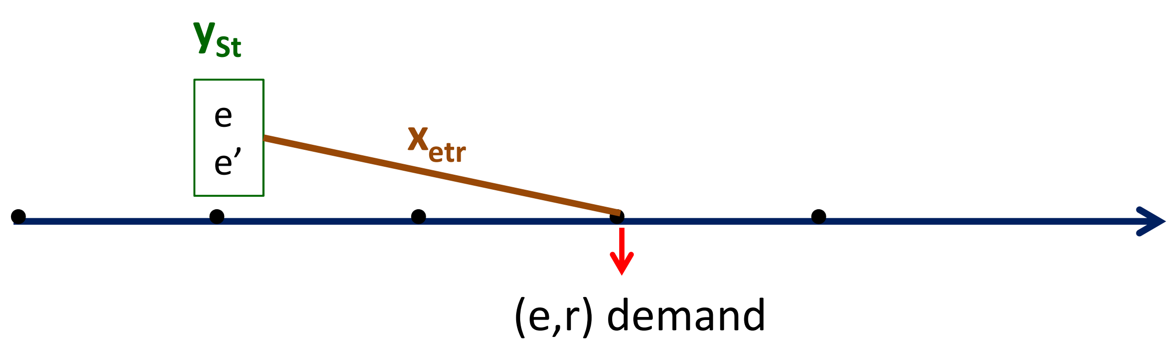
LP relaxation

min $\sum_{S,t} f(S)\, y_{St} \quad + \quad \sum_{(e,r)} h_e \sum_{t \le r} (r-t) \cdot x_{etr}$

$x_{etr} \le \sum_{S \ni e} y_{St}$ $\quad \forall$ (e,r) demand, $t \le r$

$\sum_{t \le r} x_{etr} \ge 1$ $\quad \forall$ (e,r) demand

all non-negative

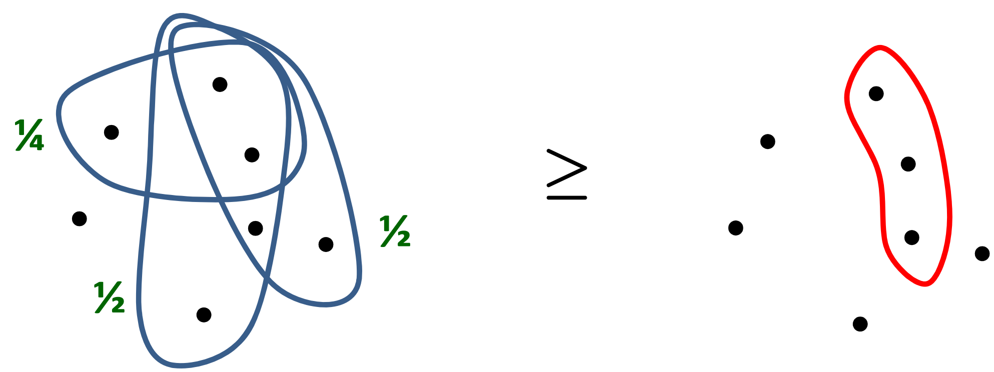


Assumptions

1. *Ordering cost f : fractionally subadditive*

   Given weighted subsets (a_i , $S_i \subseteq [N]$) with

   $\sum_{i:e \in Si} a_i \geq 1$ for all $e \in R$; then $\sum_i a_i \cdot f(S_i) \geq f(R)$ **/** α



2. *LP relaxation solvable* β **approx.**

Approximate versions suffice

Result Again

Inventory problem with

α fractional subadditive ordering cost f

β approximate LP algorithm

polynomial holding cost (degree d)

$$O\left(\alpha \, \beta \, d \; \frac{\log T}{\log \log T} \right)$$ approximation algorithm

Submodular JRP

f = submodular

- Submodular $\Rightarrow$ 1-fractionally subadditive [Feige '04]
- Solve LP via dual + ellipsoid

$\alpha = 1$
$\beta = 1$

Separation = submodular minimization

$$f(S) - \sum_{e \in S} z_{et} \geq 0 \qquad \forall S \subseteq [N], t$$

Inventory Routing

f(S) = TSP cost for visiting S

- f = 3/2-fractionally subadditive [Wolsey '80]
- Solve LP via dual + ellipsoid

  Separation = min ratio TSP

  2-approximation [Garg '02]

α = 1.5
β = 2

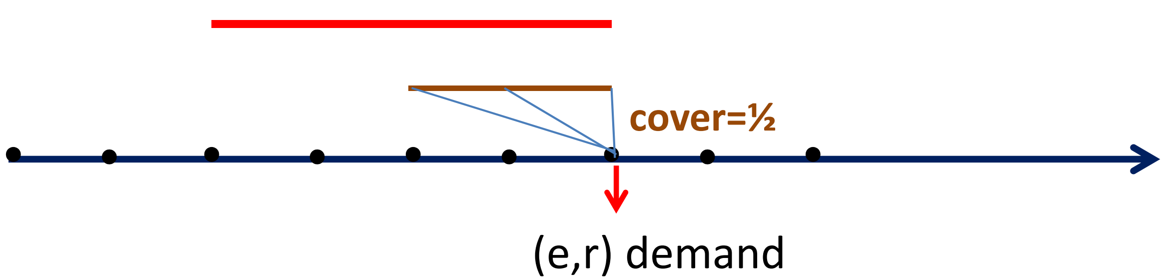
$$\mathrm{TSP}(S) - \sum_{e \in S} z_{et} \geq 0 \qquad \forall\, S \subseteq [N],\ t$$

Algorithm Outline

- Solve LP
- Find ½ completion time for each demand (e,r)
- Extend interval to power of R width

  $R = (\log T)^{½}$ for linear holding cost

- For each i = 0, 1, 2 …**L** order at integer multiples of R^i.

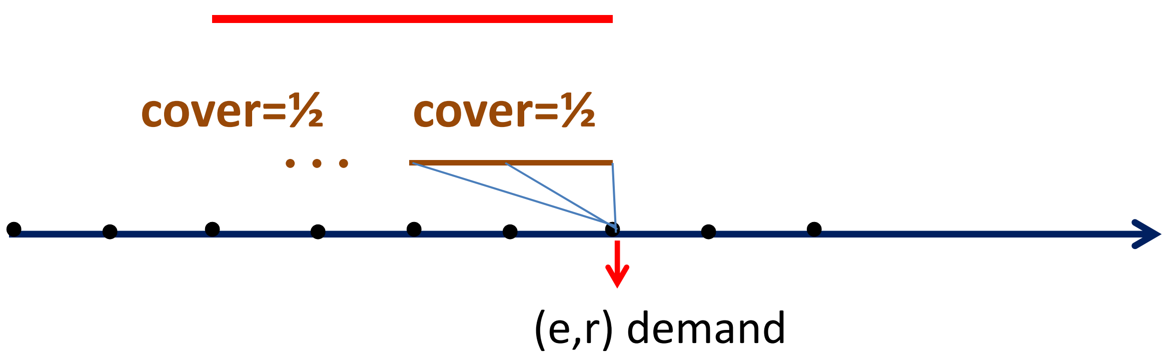


$$L = \log_R T = \frac{\log T}{\log \log T}$$

Analysis: holding cost

½ completion cost $\leq$ 2 · LP holding cost

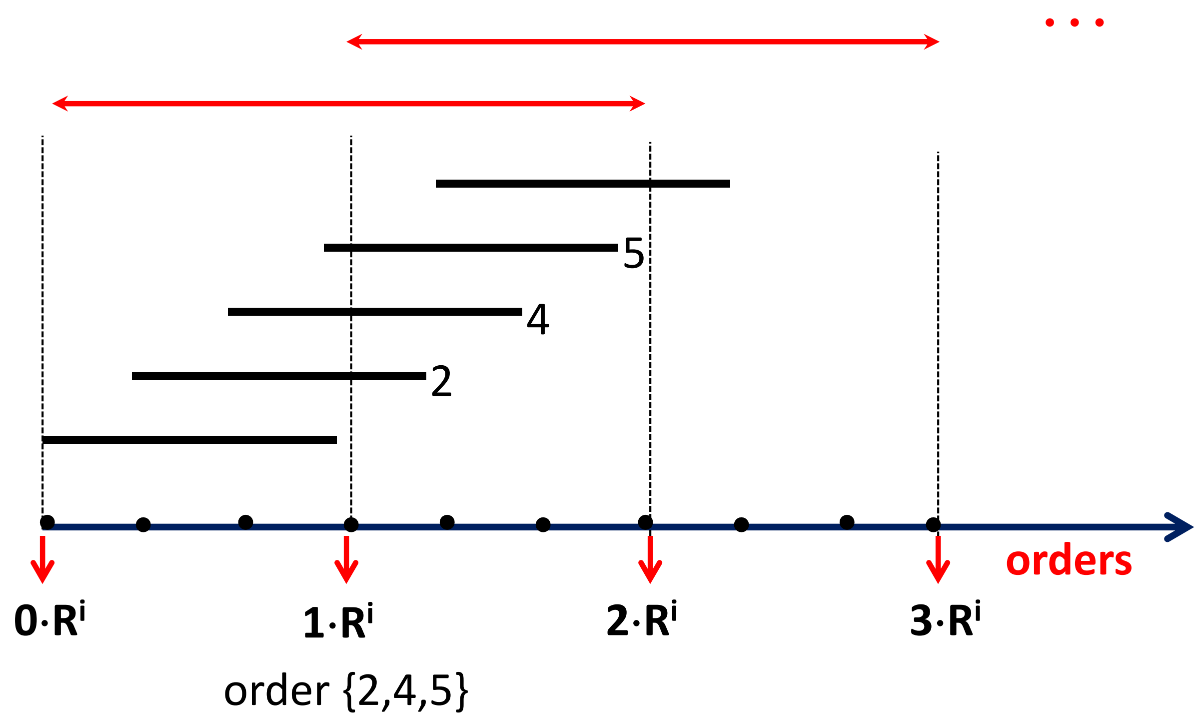
Lose factor R = (log T)^{½} since *linear*



Analysis: ordering cost

For each power of R, ordering cost $\leq 2\alpha$ LP

$$\text{overall} \leq O(\alpha) \cdot \frac{\log T}{\log \log T}$$



Summary

- Inventory problems with “complex” ordering costs

- O($\frac{\log T}{\log \log T}$) approximation ratio under

  approximate fractional subadditivity

  approximate LP

  linear holding costs

- Is there constant approximation?

Thank You!