Combinatorial algorithms for some multiflow problems and related network designs

Hiroshi Hirai
University of Tokyo

Current Trends in Combinatorial Optimization
Shonan, April 14, 2016

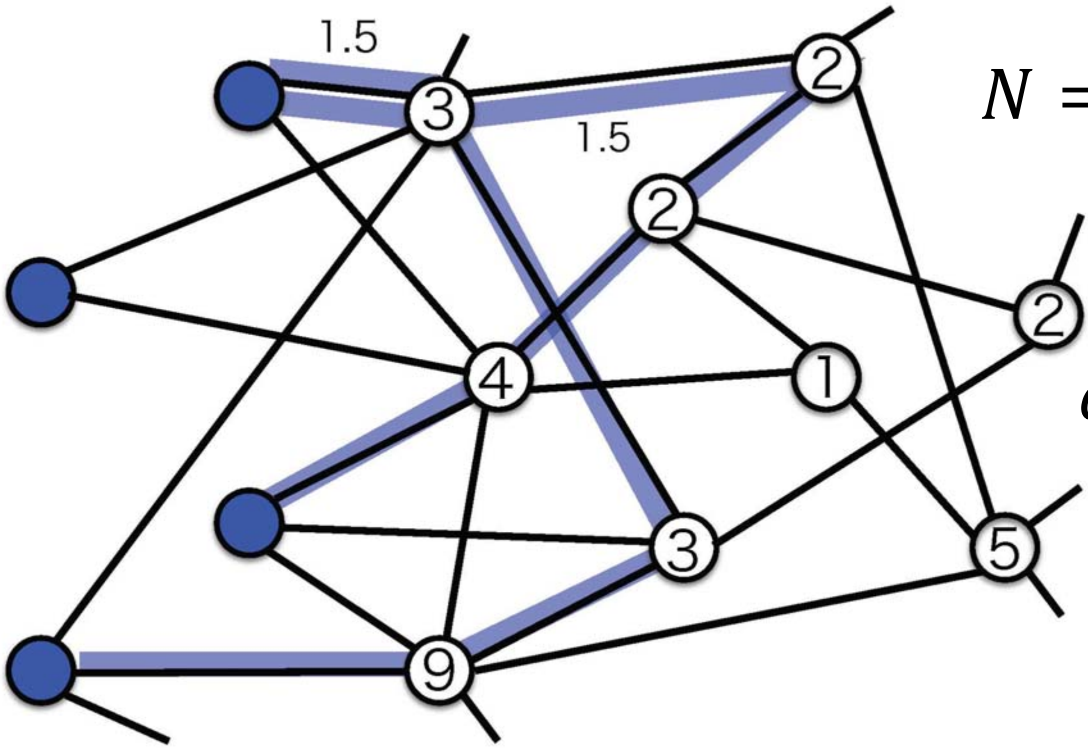
My recent research

-- Discrete Convex Analysis (Murota 1996 --)
beyond $\mathbb{Z}^n$ (H. 2012--)

-- DCA-oriented algorithm design
for multiflow, facility location, and network design

- H. Hirai: Discrete convexity and polynomial solvability in minimum 0-extension problems, SODA13, MPA15
- H. Hirai: L-extendable functions and a proximity scaling algorithm for minimum cost multiflow problem, DISOPT15
- H. Hirai: A dual descent algorithm for node-capacitated multiflow problems and its applications, 2015, arXiv.
- H. Hirai: L-convexity on graph structures, *coming soon*.

Maximum node-capacitated free multiflow



T

$N = (V, E, c, T)$

$c: V - T \to \mathbb{Z}_+$

Max. $\sum f(P)$

s.t. $f: \{T\text{–paths}\} \to \mathbb{Q}_+$

$\sum_{P: v \in P} f(P) \leq c(v) \quad (v \in V - T)$

Garg–Vazirani–Yannakakis 94

- dual of LP-relax. of node-multiway cut
- dual half-integrality → 2-approximation

Pap 08,09

- primal half-integrality (← Mader's disjoint paths)
- strongly polytime solvability (ellipsoid)

Babenko–Karzanov 08

- Combinatorial $O(\mathrm{MF}(n,m,C)n^2(\log n)^2 \log C)$-time algo.

Result

Combinatorial $O(m \log |T| \, \mathrm{MSF}(n, m, 1))$–time algo.
to find half-integral primal & dual opt.

$\mathrm{MSF}(n, m, h)$: time of solving max. submodular flow
h: time of computing exchange capacity

Fujishige–Zhang 92: $O(n^3 h)$–time algo. for max. subflow.
--- $O(mn^3 \log |T|)$–time algo.
--- for 2–approx. of node–multiway cut

Related work:
Chekuri–Madan SODA16: $(2+\mathcal{E})$–approx. $\tilde{O}(nm/\mathcal{E})$–time algo.
for node–multiway cut

basic idea --- solve dual first, recover primal, combinatorially

LP-dual:

$$\min. \sum_{i \in V-T} c(i)w(i)$$

$$\text{s.t.} \sum_{i \in P-T} w(i) \geq 1 \quad (P\text{: } T\text{-path})$$

$$w(i) \geq 0 \; (i \in V - T)$$

($w(i) \in \{0,1\}$ → node-multiway cut)

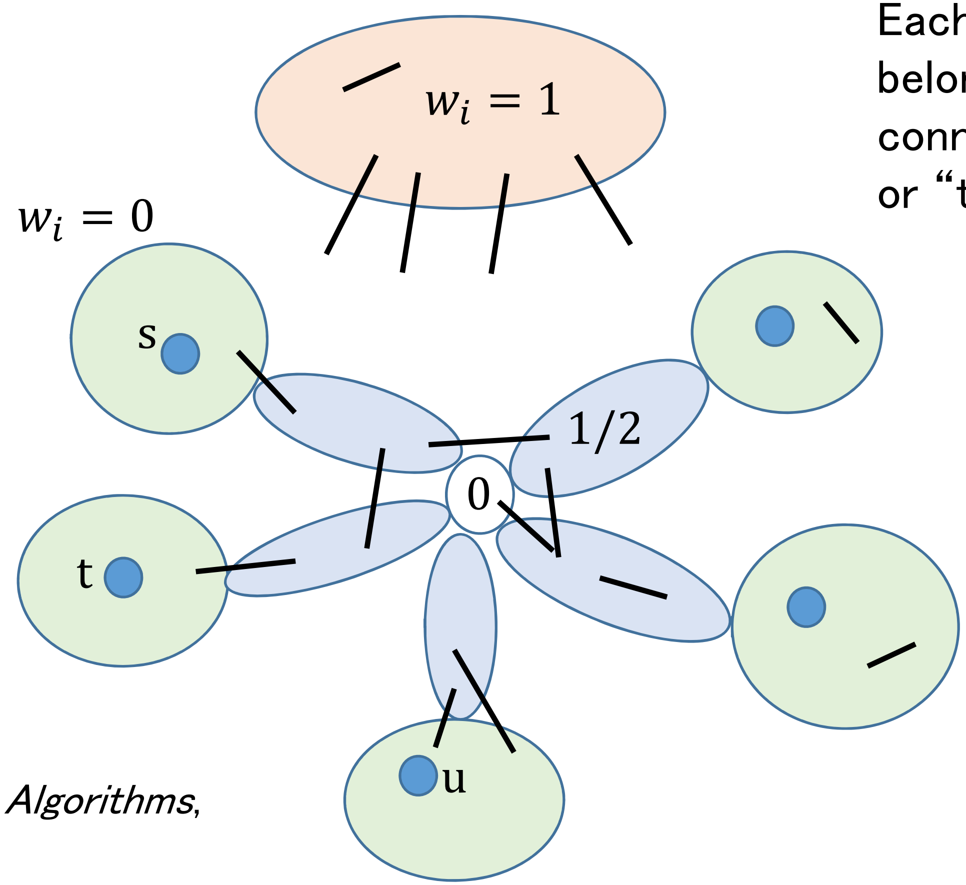
Garg–Vazirani–Yannakakis half-integrality

$$\min. \sum_{i \in V-T} c(i)w(i)$$

s.t. partitions of V:

$|T|$ green parts
$|T|$ blue parts

Each edge
belongs to same part, or
connects “adjacent” parts
or “top” part



Each green part
contains
exactly one terminal

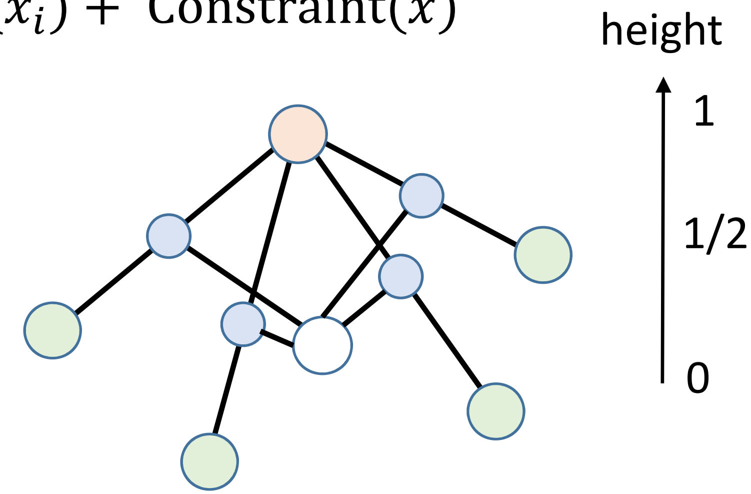
Vazirani: *Approximation Algorithms*,
Section 19.3

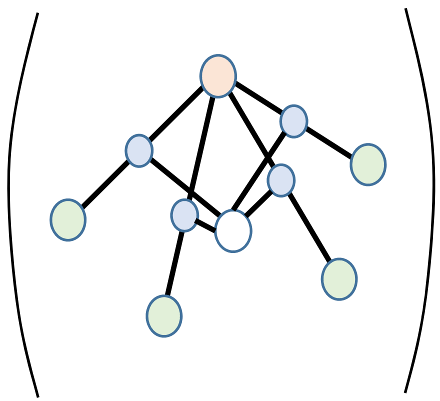
Submodular interpretation

$$\text{Min. } g(x) := \sum_i \text{height}(x_i) + \text{Constraint}(x)$$

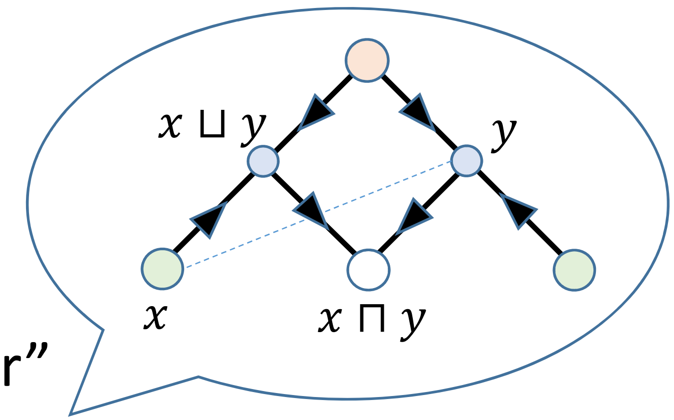
$\in \{0, \infty\}$

s.t. $x_i \in$ $(i \in V)$



$g:$  $^n \longrightarrow \mathbb{Q} \cup \{+\infty\}$

is “submodular”



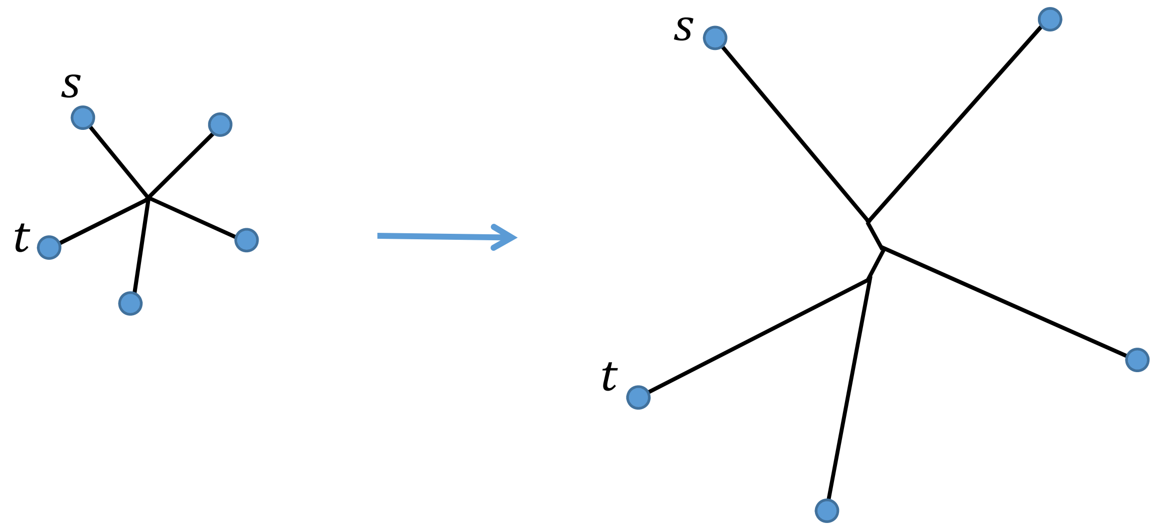
$$g(x) + g(y) \geq g(x \sqcap y) + g(x \sqcup y)$$

I do not know
how to solve this SFM & how to recover primal opt,
directly & combinatorially

$$\text{Max.} \quad \sum f(P) \xrightarrow{\text{perturb}} \sum \mu(s_P, t_P) f(P) - \overbrace{\sum f(e)}^{\text{edge-cost}}$$

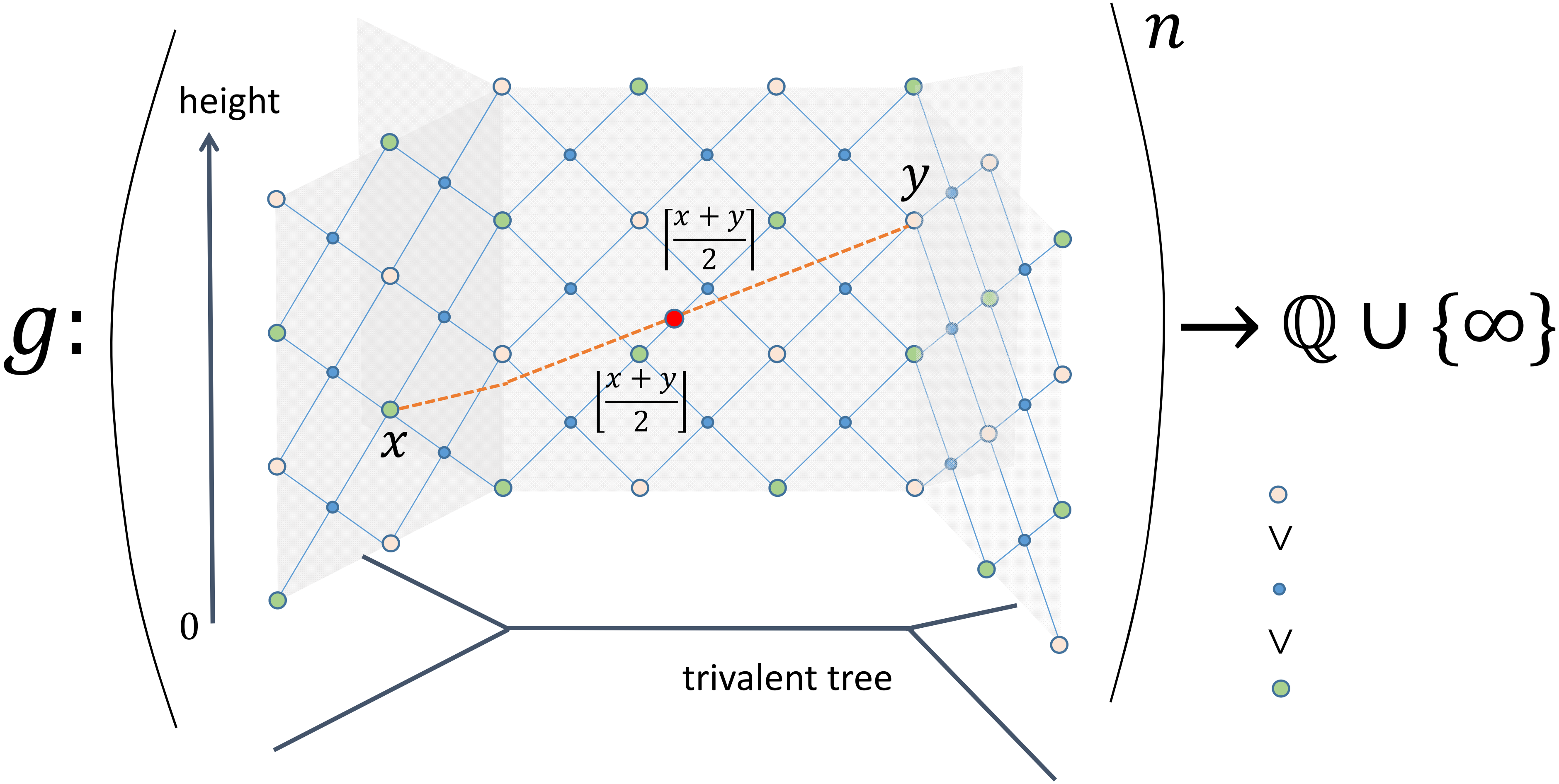
$$\text{s.t.} \quad \text{multiflow } f$$

$\mu: \binom{T}{2} \to \mathbb{Z}_+$ trivalent tree metric



-- half-integrality preserved
-- opt. recovered by perturbed opt.

(Perturbed) dual objective is “L-convex”

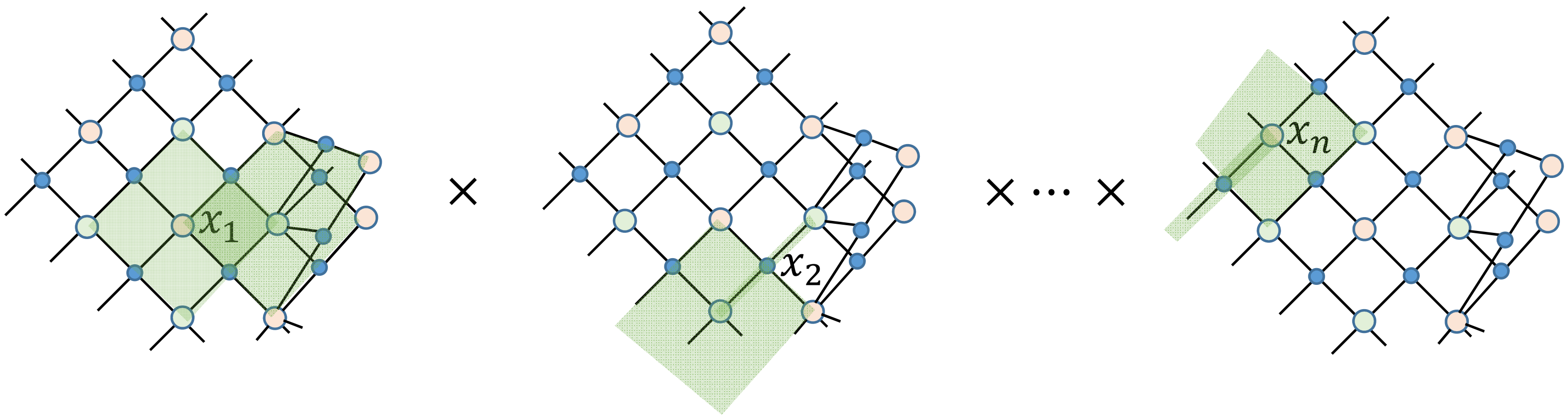


$$g(x) + g(y) \geq g\left(\left\lceil \frac{x+y}{2} \right\rceil\right) + g\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right)$$

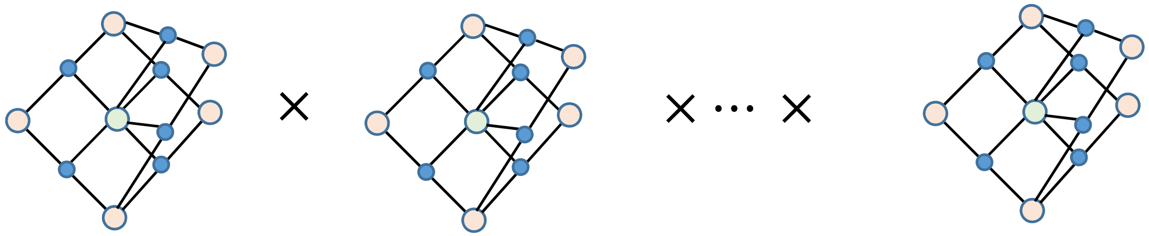
Favati–Tardella, Murota, Fujishige for $\mathbb{Z}^n$, H. beyond $\mathbb{Z}^n$

A framework for minimizing L−conv. func.

Local opt = Global opt → Steepest Descent Algorithm



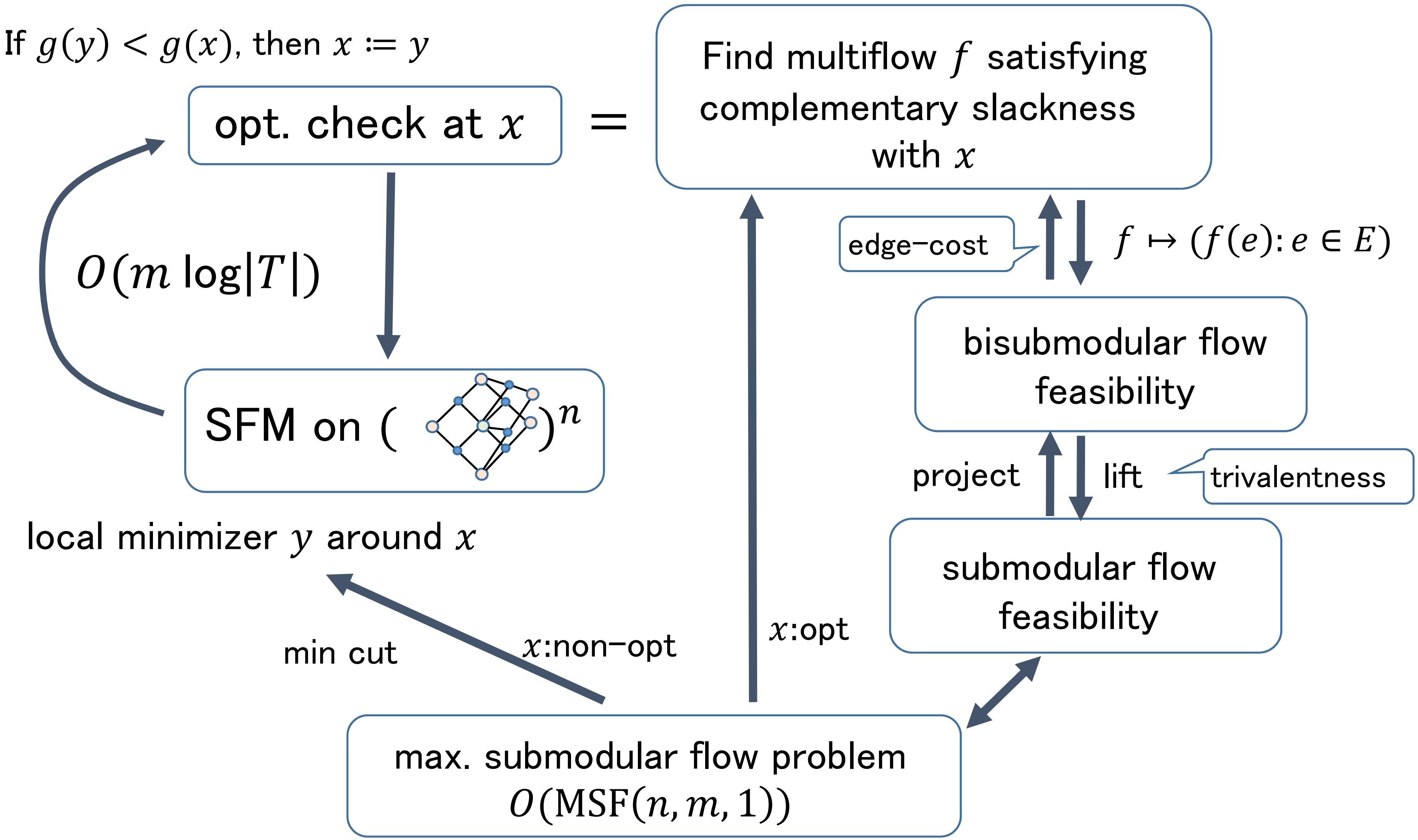
Local problem = SFM on



iteration of SDA $\leq$ l_∞−diameter of domain

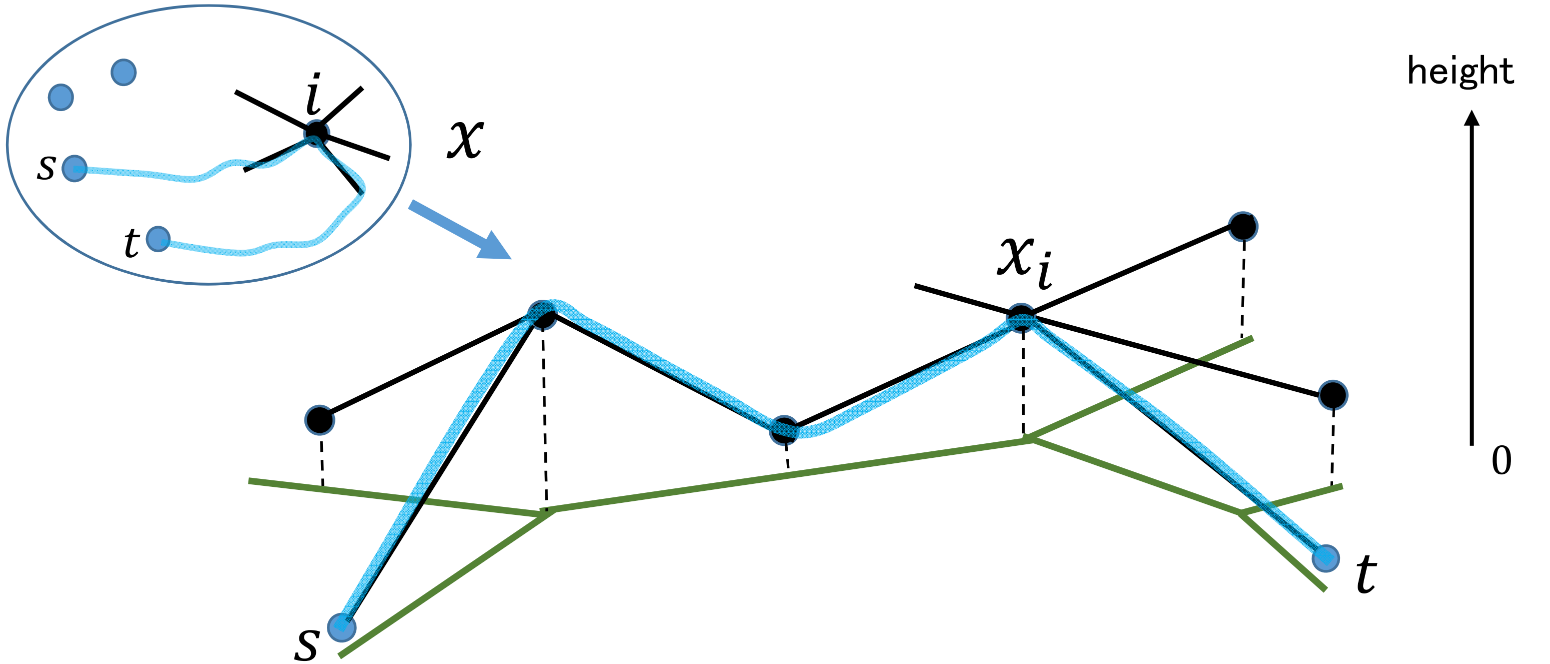
$= O(m \log|T|)$ (in our case)

Algorithm


If $g(y) < g(x)$, then $x := y$
opt. check at x
=
Find multiflow f satisfying complementary slackness with x
$O(m \log|T|)$
edge−cost
$f \mapsto (f(e): e \in E)$
bisubmodular flow feasibility
SFM on ($)^n$
project
lift
trivalentness
submodular flow feasibility
local minimizer y around x
min cut
x:non−opt
x:opt
max. submodular flow problem
$O(\mathrm{MSF}(n, m, 1))$

Why (bi)submodular flow ?

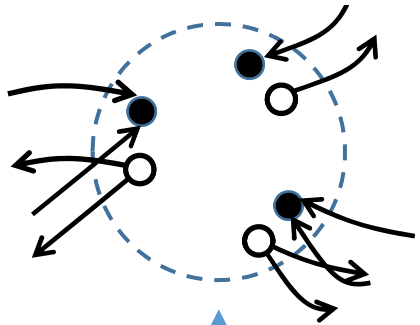
$N = (V, E, c, T)$

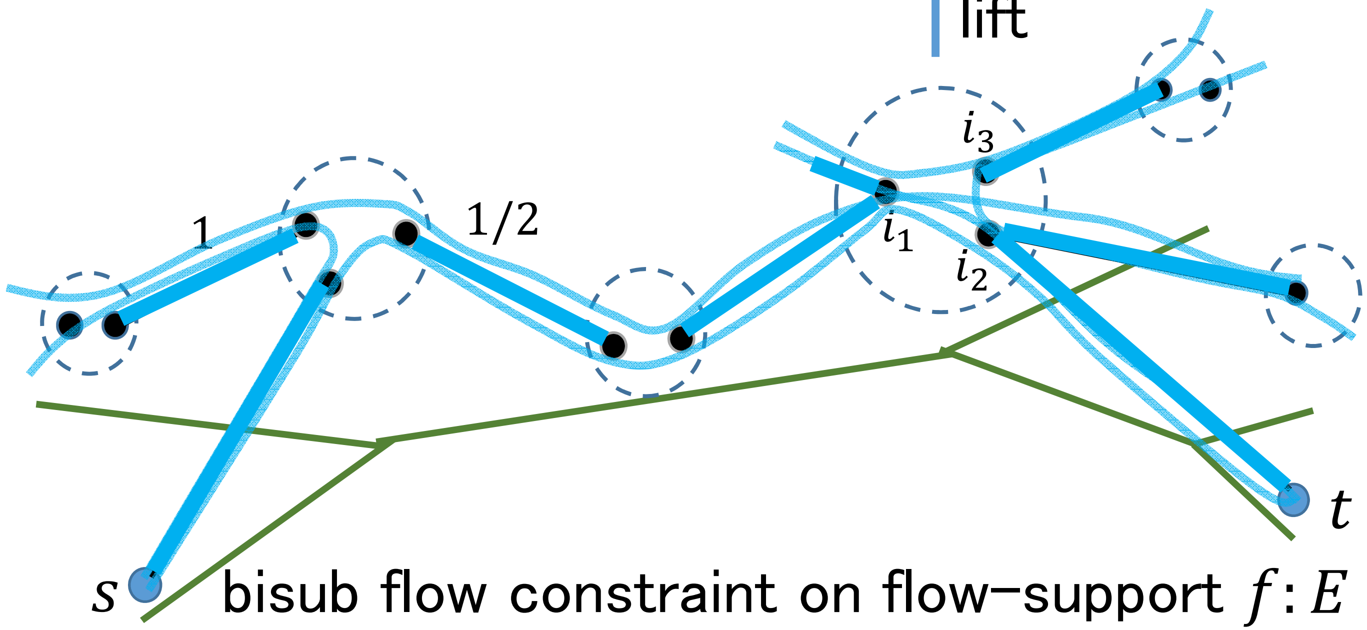


Complementary slackness:

1. $f(i) = c(i)$ if $\mathrm{height}(x_i) > 0$
2. $f(ij) = 0$ if x_i, x_j are "close"
3. f induces "geodesic" flow by x

submodular flow
constraint
of “disjoint” 6−dim submo.
---> $h = O(1)$



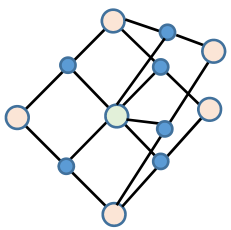
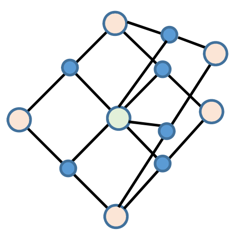
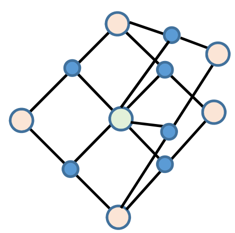


bisub flow constraint on flow−support $f: E \to \mathbb{R}_+$

1. $f(\delta(i_1)) + f(\delta(i_2)) + f(\delta(i_3)) \underset{\le}{=} 2c(i)$

3. $f(\delta(i_1)) + f(\delta(i_2)) \ge f(\delta(i_3))$

Summary, remark, question

- First combinatorial strong polytime algo. for max. node-cap. free mulitflow
- DCA idea brings combin. polytime algo. for another mincost multilfow problem [H.15]
- ?? more direct combinatorial algorithm without perturbation
- ?? SFM on  ×  × ··· × 

Thank you for your attention