Coloring low-discrepancy hypergraphs, Weak Polymorphisms, and Promise Constraint Satisfaction

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2-coloring graphs and hypergraphs

Given a graph G, it is easy to tell if it is 2-colorable (bipartite).

Given a 3-uniform hypergraph, NP-hard to tell if it is 2-colorable (with no monochromatic hyperedges).

Will later muse: What "causes" this dramatic change in computational complexity?

2-coloring: a finer dichotomy

Fact (nice exercise): Given a 2t-uniform hypergraph that is promised to admit a *perfectly balanced 2-coloring*, one can *efficiently* find a 2-coloring without monochromatic hyperedges.

(2-coloring graphs is special case when $t=1$)

What if there is an almost-balanced 2-coloring (hypergraph has low discrepancy)?

Theorem [Austrin-G.-Hastad'14]:. $\forall t \geq 1$, given a (2t+1)-uniform hypergh, it is NP-hard to distinguish between following cases:

- (Yes): there is a near-balanced coloring (with discrepancy 1),
- (No): every 2-coloring leaves a monochromatic *hyperedge*

This Talk

Two directions in which AGH can be strengthened/generalized

1. Stronger hardness for hypergraphs with nice colorings
2. Study broader class of "promise problems"

   (that generalize coloring low-discrepancy hypergraphs)

Stronger coloring hardness I

Hardness of *approximate* hypergraph coloring
[G.-Håstad-Sudan'00; Dinur-Regev-Smyth'03]
For $r \geq 3$, it is NP-hard to color a 2-colorable r-uniform hypergraph with 10^{10} colors.

<u>Comment</u>: Recent developments (spurred by low-degree long code), show hardness of coloring 2-colorable hypergraphs with $\exp((\log n)^{1/10})$ colors [Dinur-G.'13; G.-Harsha-Hastad-Srinivasan-Varma'14; Khot-Saket'14, Huang'15]

<u>Theorem</u>: [G.-Lee'15] Approximate hypergraph coloring is hard even when a *near-balanced 2-coloring* exists: $\forall\, t \geq 2$,

Given a 2t-uniform hypergraph admitting a near-balanced 2-coloring ($\leq t+1$ vertices of a color in each hyperedge),

it is NP-hard to color it with 10^{10} colors.

Stronger coloring hardness II

What if we are promised something stronger than low-discrepancy?

Eg., what if a r-uniform hypergraph is $(r+1)$-*strongly colorable*?

- ℓ-strongly colorable: vertex coloring with ℓ colors so that every hyperedge has *all vertices of different colors*
- ℓ-strongly colorable for $\ell \leq 2r-2$ implies 2-colorability

<u>Theorem</u>: [Brakensiek-G.'16a]
Given a $\lceil \frac{3r}{2} \rceil$-strongly colorable r-uniform hypergraph, it is NP-hard to 2-color it (with no monochromatic edges)

We conjecture that similar hardness holds for $(r+1)$-strongly colorable case (which will imply the AGH hardness for low-discrepancy hypergraphs)

This Talk

Two directions in which AGH can be strengthened/generalized

1. Stronger hardness for hypergraphs with nice colorings

2. Study broader class of "promise problems"

   (that generalize coloring low-discrepancy hypergraphs)

Back to Slide 1

Given a graph G, it is easy to tell if it is 2-colorable (bipartite).

- Collection of $\neq(x,y)$ constraints

Given a 3-uniform hypergraph, NP-hard to tell if it is 2-colorable (with no monochromatic hyperedges).

- Collection of Not-all-Equal(x,y,z) constraints

What "causes" this dramatic change in computational complexity?

Operations preserving relations

$f : \{0,1\}^L \to \{0,1\}$ preserves a relation $R \subseteq \{0,1\}^k$ if
$\forall \left(a_1^i, a_2^i, \dots, a_k^i\right) \in R, i = 1,2, \dots, L,$
$\left(f(a_1^1, a_1^2, \dots, a_1^L), \dots, f\left(a_k^1, a_k^2, \dots, a_k^L\right) \right) \in R$

Imagine $k \times L$ matrix whose *columns* belong to R.
Then applying f *row-wise* yields a k-tuple in R.

$f(\ 0\ \ 1\ \ 1\ \ \dots\ \ 0\ ) = b_1$
$f(\ 1\ \ 0\ \ 0\ \ \dots\ \ 1\ ) = b_2$

$R_2 = \{(0,1),(1,0)\}$ (2-coloring)

Which f preserve R_2? Antipodal fns

$f(\ 0\ \ 1\ \ 1\ \ \dots\ \ 0\ ) = b_1$
$f(\ 0\ \ 0\ \ 1\ \ \dots\ \ 1\ ) = b_2$
$f(\ 1\ \ 0\ \ 0\ \ \dots\ \ 1\ ) = b_3$

$R_3 = \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$
(2-coloring 3-hypergraphs)

Which f preserve R_3?

Dictator functions

Polymorphisms & complexity distinction?

Polymorphisms of R (Pol(R)) = $\{ f \mid f$ preserves $R\}$

Easy vs hard explained by:

- For 2-coloring graphs, non-trivial polymorphisms exist (Majority, Odd Parity, any antipodal function)
- For 2-coloring 3-hypergraphs, there are only trivial (dictatorial) polymorphisms

Constraint Satisfaction Problem (CSP)

$\Gamma = \{R_1, R_2, \ldots, R_s\}$ finite set of relations over $\{0,1\}$

(more generally, any finite domain D)

CSP(Γ) instance given by (V, C):

- V set of variables
- C set of constraints of form (τ, P) where
  $P \in \Gamma$ & τ is a k-tuple of variables ($k = \text{arity}(P)$)

Goal: Is there an assignment $\sigma: V \to D$ that satisfies all constraints in C, i.e., $\big(\sigma(\tau_1), \ldots, \sigma(\tau_k)\big) \in P\ \forall (\tau, P) \in C$?

2-colorability of graphs: CSP($\neq$)

2-colorability of 3-hypergraphs: CSP(NAE_3)

Boolean CSP dichotomy theorem [Schaefer 1978]
For every finite set Γ of relations over $\{0,1\}$,
CSP(Γ) is in P or NP-complete.

CSP(Γ) is in P if:

i. Every relation in Γ is 0-valid
ii. Every relation in Γ is 1-valid
iii. Every relation in Γ is a 2CNF
iv. Every relation in Γ is affine
v. Every relation in Γ is a conjunction of Horn clauses
vi. Every relation in Γ is a conjunction of dual Horn clauses

Otherwise CSP(Γ) is NP-complete

Contemporary polymorphic view

Theorem: A Boolean CSP(Γ) is

- polytime solvable if Pol(Γ) is non-trivial (contains a non-dictator), and
- NP-complete if Pol(Γ) is trivial (contains only dictators).

Polymorphisms in tractable cases of Schaefer's theorem:

- Constant 0 or 1 function (trivial cases)
- Majority(x,y,z) (2SAT)
- AND(x,y) (Horn SAT)
- OR(x,y) (dual Horn SAT)
- $x \oplus y \oplus z$ (affine equations)

Thm: CSP(Γ) is in P if Pol(Γ) contains 0,1, Majority (of odd arity), AND, OR, parity (of odd arity); otherwise it is NP-complete

CSP dichotomy conjecture

Feder-Vardi conjecture (1998): For every Γ over an *arbitrary* finite domain, CSP(Γ) is in P or NP-complete.

Algebraic dichotomy conjecture (Bulatov-Jeavons-Krokhin'05): CSP(Γ) is in P if Pol(Γ) contains non-dictatorial functions; otherwise CSP(Γ) is NP-complete.

Polymorphisms are the correct tool:

- Galois correspondence: Pol(Γ) $\subseteq$ Pol(Γ') $\Rightarrow$ Γ' reduces to Γ
- CSPs with same polymorphisms have identical complexity
- If Pol(Γ) is trivial, then CSP(Γ) is NP-complete.

Low-discrepancy 2-coloring as "promise" CSP

Given a 2t-uniform hypergraph that is promised to admit a *perfectly balanced 2-coloring*, one can *efficiently* find a 2-coloring that avoids monochromatic edges.

$P = \{\, x \in \{0,1\}^{2t} \mid \mathrm{wt}(x) = t\}$ $\qquad Q = \{0,1\}^{2t} \setminus \{0^{2t}, 1^{2t}\}$

Given a satisfiable CSP(P) instance, can find an assignment satisfying it as a CSP(Q) instance.

Polymorphic explanation?

- ***Weak polymorphism*** $f \in \mathrm{Pol}(P, Q)$ is a function mapping inputs in P to output in Q

For above pair, Majority$_L$ for odd L is a weak polymorphism

Promise CSP view of AGH theorem

Given a (2t+1)-uniform hypergh, NP-hard to distinguish whether
- (Yes): there is a near-balanced coloring (with discrepancy 1),
- (No): every 2-coloring leaves a monochromatic *hyperedge*

$P = \{ x \in \{0,1\}^{2t+1} \mid \text{wt}(x) \in \{t, t+1\}\}$ $\quad Q = \{0,1\}^{2t+1} \setminus \{0^{2t+1}, 1^{2t+1}\}$

Thm restatement: Given a satisfiable CSP(P) instance, it is NP-hard to find an assignment satisfying it as a CSP(Q) instance.

But there are *non-dictatorial weak polymorphisms*
- Majority$_m$ $\in$ Pol(P, Q) for odd $m \leq 2t - 1$

Reason behind AGH theorem:
- If $f \in$ Pol(P, Q) then f is a $(2t-1)$-junta*
- Proof based on elementary combinatorial arguments
- Hardness via reduction from Label Cover

Promise CSP dichotomy?

For two predicates $P \subseteq Q \subseteq \{0,1\}^k$, PCSP($P, Q$) is the problem of telling if (i) an instance is satisfiable as a CSP(P) instance, or
(ii) it is unsatisfiable even as a CSP(Q) instance.

Captures notorious problems such as approximate graph coloring:

- $P = \{(1,2),(2,3),(3,1),(2,1),(3,2),(1,3)\}$
- $Q = \{(i,j) \mid i,j \in \{1,2,,\ldots,10\}; i \neq j\}$

PCSP(Γ) defined similarly for a finite set $\Gamma = \{\,(P_1,Q_1),\ldots,(P_s,Q_s)\}$ of relation-pairs, $P_i \subseteq Q_i$

Questions driving this research agenda:
Is every PCSP(Γ) in P or NP-hard?
Do weak polymorphisms govern this dichotomy?
Can one characterize the tractable cases?

Harder than CSP dichotomy conjecture! Focus on Boolean domain.

Algorithms for PCSPs

Recall: $\exists$ efficient algorithm for PCSP(P, Q) for

$P = \{\, x \in \{0,1\}^{2t} \mid \mathrm{wt}(x) = t\}$ $\qquad Q = \{0,1\}^{2t} \setminus \{0^{2t}, 1^{2t}\}$

(Weak) polymorphism behind algo: Majority of odd # inputs

Turns out there there are algorithms based on
new polymorphisms in PCSP setting…

A new weak polymorphism

Proposition (Brakensiek-G.'16) PCSP(P, Q) is tractable for

$$P = \{\, x \in \{0,1\}^k \mid \mathrm{wt}(x) = a\} \qquad Q = \{0,1\}^k \setminus \{0^k, 1^k\}$$

(for any $a \in \{1,2, \dots, k-1\}$)

Hypergraph $H = (V, E)$ with integer a_e, $\forall e \in E, 0 < a_e < |e|$; if $\exists$ a 2-coloring with exactly a_e vertices colored Red in each e, then can efficiently 2-color it without monochromatic edges.

Algorithm is based on linear programming

Underlying weak polymorphism: *Alternating threshold* (of odd arity)

$$AT_L(z_1, z_2, \dots, z_L) = 1(z_1 - z_2 + z_3 - z_4 + \ \dots - z_{L-1} + z_L > 0)$$

Symmetric CSPs

A relation $P \subseteq \{0,1\}^k$ is symmetric if membership $x \in P$ only depends on the Hamming weight of x

CSP(Γ)/PCSP(Γ) is symmetric if all relations in Γ are symmetric.

Natural subclass (k-SAT, NAE k-SAT, discrepancy, etc. are symmetric)
Horn SAT is *not* symmetric

Partial Dichotomy Theorem for promise CSPs [Brakensiek-G.'16]
Let PCSP(Γ) be any Boolean promise CSP with symmetric relations (allowing negations). Then PCSP(Γ) is in P or NP-hard.

Tractable cases

Thm [Brakensiek-G.'16] Let Γ be a set of symmetric relation pairs (including Boolean negation).

Then PCPS(Γ) is in P if weak polymorphisms of Γ contain:

i. Parity of L variables for all odd L, or
ii. Majority of L variables for all odd L, or
iii. Alternating Threshold of L variables for odd L, or
iv. the "anti"-version of any of the above for all odd L.

Otherwise PCSP(Γ) is NP-hard.

Algorithms:

- Case i: Gaussian elimination over F_2
- Cases ii,iii: Linear programming
- Work for general, non-symmetric case as well

A word about hardness side

Structural classification of weak polymorphisms:
If Γ doesn't admit $Parity_{L_1}$,Maj_{L_2}, AT_{L_3} as a weak polymorphism for some odd L_1, L_2, L_3, (or their "antis" for some odd arity), then there exists $C(\Gamma) < \infty$ such that *every* weak polymorphism $f\colon \{0,1\}^m \to \{0,1\}$ of Γ satisfies:

- $\exists\ S \subset \{1,2,\ldots,m\}$ of size $C(\Gamma)$ s.t.
  $f(x) = f(000\ldots00)$ if $x_i = 0\ \forall i \in S$

While we don't get a dictator/junta,
this interfaces well with a Label Cover reduction.

Summary

Promise CSPs capture several interesting problems, including variants of graph/hypergraph coloring

Viewpoint has led to new hardness results for coloring

- 6-coloring 4-colorable graphs is NP-hard
- new NP-hardness proof that 4-coloring 3-colorable graphs (already known in [Khanna-Linial-Safra'93, G.-Khanna'00]

Partial dichotomy theorem: Every symmetric Boolean promise CSP (allowing negations) is either in P or NP-hard

➢ Road beyond symmetric case seems difficult but exciting
  - Have to cope with further algorithms & polymorphisms, and work with less structure for hardness.