A Solution to the Random Assignment Problem with a Matroidal Family of Goods

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NII Shonan Meeting on Current Trends in Combinatorial Optimization
April 11–14, 2016

Joint work with
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E: a finite set of **objects**

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$\mathcal{B} \subseteq 2^E$: a family of available (feasible) sets of objects that forms **a family of bases of a matroid** on E

Without money we consider how to choose one base B from among $\mathcal{B}$ and allocate the goods in B to agents **in an efficient and fair manner**.

(The case when $\mathcal{B}$ consists of a single base has been considered in the literature.)

$\rightarrow$

$(E, \mathcal{B})$: a matroid with its rank function $\rho : 2^E \to \mathbb{Z}_{\geq 0}$

The **base polytope** of the matroid (the convex hull of all the characteristic vectors χ_B of bases $B \in \mathcal{B}$):

$$\mathrm{B}(\rho) = \{x \in \mathbb{R}^E \mid \forall X \subset E : x(X) \leq \rho(X),\ x(E) = \rho(E)\},$$

where for any $X \subseteq E$ we define $x(X) = \sum_{e \in X} x(e)$.

The **submodular polyhedron** associated with ρ:

$$\mathrm{P}(\rho) = \{x \in \mathbb{R}^E \mid \forall X \subseteq E : x(X) \leq \rho(X)\}$$

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The **submodular polyhedron** associated with ρ:

$$\mathrm{P}(\rho) = \{x \in \mathbb{R}^E \mid \forall X \subseteq E : x(X) \leq \rho(X)\}$$

Given a vector $x \in \mathrm{P}(\rho)$ a subset X of E is called *tight* for x if we have $x(X) = \rho(X)$.

sat(x): a unique maximal tight set for x

$$\mathrm{sat}(x) = \{e \in E \mid \forall \alpha > 0 : x + \alpha\chi_e \notin \mathrm{P}(\rho)\}$$

(Matroid $(E, \mathcal{B})$ is often denoted by (E, ρ) as well.)

$\rightarrow$

For each $i \in N$ let **agent i's preference** be given by

$$L^i : \ e^i_1 \succ_i e^i_2 \succ_i \cdots \succ_i e^i_m,$$

where $\{e^i_1, e^i_2, \cdots, e^i_m\} = E$.

$\mathcal{L}$: the profile of preferences L^i $(i \in N)$

e^i_1: the **top** (most favorite) **good** of agent $i \in N$

Define a nonnegative integral vector $b(\mathcal{L}) \in \mathbb{Z}^E_{\geq 0}$ by

$$b(\mathcal{L}) = \sum_{i \in N} \chi_{e^i_1},$$

where we may have $e^i_1 = e^j_1$ for distinct $i, j \in N$.

$\rightarrow$

An $N \times E$ matrix $P = (P(i,e) \mid i \in N,\ e \in E)$ is called a *random assignment* if it satisfies

1. $P(i,e) \geq 0$ for all $i \in N$ and $e \in E$,
2. regarding each ith row P_i of P as a vector in $\mathbb{R}^E_{\geq 0}$, we have

$$x^*_P \equiv \sum_{i \in N} P_i \in \mathrm{B}(\rho).$$

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$$x^*_P \equiv \sum_{i \in N} P_i \in \mathrm{B}(\rho).$$

First, we consider $\mathrm{B}(\rho)$ as a set of **divisible** goods and find an allocation of the divisible goods in an efficient and fair manner.

$\rightarrow$

Random_Assignment

Input: Preferences $\mathcal{L} = (L^i \mid i \in N)$ and a matroid (E, ρ) with $\rho(E) \le |N| (= n)$.

Output: A random assignment matrix $P \in \mathbb{R}_{\ge 0}^{N \times E}$ and a base $x_P^* \in \mathrm{B}(\rho)$.

Step 0: For each $i \in N$ put $x^i \leftarrow \mathbf{0} \in \mathbb{R}^E$ (the zero vector) and put $S_0 \leftarrow \emptyset$, $p \leftarrow 1$, and $x^* \leftarrow \mathbf{0}$.

Step 1: For current (updated) $\mathcal{L} = (L^i \mid i \in N)$ compute

$$\lambda^* = \max\{\lambda \ge 0 \mid x^* + \lambda b(\mathcal{L}) \in \mathrm{P}(\rho)\}.$$

For each $i \in N$ put $x^i \leftarrow x^i + \lambda^* \chi_{e_1^i}$.
Put $x^* \leftarrow x^* + \lambda^* b(\mathcal{L})$ and $S_p \leftarrow \mathrm{sat}(x^*)$ for $x^* \in \mathrm{P}(\rho)$.

Step 2: Put $T \leftarrow S_p \setminus S_{p-1}$.
Remove all elements of T and update L^i $(i \in N)$.

Step 3: If $\rho(S_p) < \rho(E)$, then put $p \leftarrow p + 1$ and go to Step 1.
Otherwise put $P(i, e) \leftarrow x^i(e)$ for all $i \in N$ and $e \in E$.
Return P and $x_P^* = x^*$.

Note that $x_P^* = x^*$ and for each agent $i \in N$ the ith row sum of P is equal to $\rho(E)/|N|$.

$\rightarrow$

Example 1:

$N = \{1, 2, 3, 4\}, \; E = \{a, b, c, d\}$

Consider a uniform matroid $\mathbf{M} = (E, \mathcal{B})$ of rank two.

Preferences of all agents are given by

$i \in N$	preference L^i
1	$a \succ_1 b \succ_1 c \succ_1 d$
2	$a \succ_2 c \succ_2 b \succ_2 d$
3	$a \succ_3 c \succ_3 d \succ_3 b$
4	$b \succ_4 a \succ_4 d \succ_4 c$

$\rightarrow$

By Random_Assignment we have

$i \in N$	preference L^i
1	$a \succ_1 b \succ_1 c \succ_1 d$
2	$a \succ_2 c \succ_2 b \succ_2 d$
3	$a \succ_3 c \succ_3 d \succ_3 b$
4	$b \succ_4 a \succ_4 d \succ_4 c$

$$b(\mathcal{L}) = \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{pmatrix} 3 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$$S_1 = \{a\}$$

$$\lambda^* = 1/3 \text{ for } p = 1$$

$$P = \begin{array}{c|cccc} & a & b & c & d \\ \hline 1 & 1/3 & 1/6 & 0 & 0 \\ 2 & 1/3 & 0 & 1/6 & 0 \\ 3 & 1/3 & 0 & 1/6 & 0 \\ 4 & 0 & 1/3 + 1/6 & 0 & 0 \end{array}$$

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By Random_Assignment we have

$i \in N$	preference L^i
1	$a \succ_1 b \succ_1 c \succ_1 d$
2	$a \succ_2 c \succ_2 b \succ_2 d$
3	$a \succ_3 c \succ_3 d \succ_3 b$
4	$b \succ_4 a \succ_4 d \succ_4 c$

$$b(\mathcal{L}) = (0, 2, 2, 0)$$
$$S_1 = \{a\}, \quad S_2 = \{a, b, c, d\}$$
$$\lambda^* = 1/6 \text{ for } p = 2$$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1/3 & 1/6 & 0 & 0 \\ 1/3 & 0 & 1/6 & 0 \\ 1/3 & 0 & 1/6 & 0 \\ 0 & 1/3 + 1/6 & 0 & 0 \end{pmatrix} \end{array}$$

$$x_P^* = (1, 2/3, 1/3, 0)$$

$\rightarrow$

Example 2: $E = \{a, b, c, d\}$

$$\mathcal{B} = \{X \mid X \subset E,\ |X| = 2,\ X \neq \{a, b\}\}$$

This is a graphic matroid, which is represented by

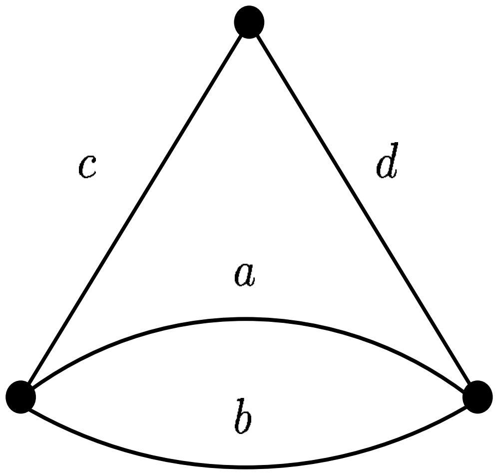

Figure 1: A graph with edge set $E = \{a, b, c, d\}$.

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1	$a \succ_1 b \succ_1 c \succ_1 d$
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3	$a \succ_3 c \succ_3 d \succ_3 b$
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$b(\mathcal{L}) = (3, 1, 0, 0), \quad (0, 0, 3, 1)$

$S_1 = \{a, b\}, \qquad S_2 = \{a, b, c, d\}$

$\lambda^* = 1/4$ **for** $p = 1$ **and** $\lambda^* = 1/4$ **for** $p = 2$.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1/4 & 0 & 1/4 & 0 \\ 1/4 & 0 & 1/4 & 0 \\ 1/4 & 0 & 1/4 & 0 \\ 0 & 1/4 & 0 & 1/4 \end{pmatrix} \end{array}$$

$x^*_P = (3/4, 1/4, 3/4, 1/4)$.

$\rightarrow$

Let P and Q be random assignments.
For each agent $i \in N$ with preference relation $\succ_i$ given by $e^i_1 \succ_i \cdots \succ_i e^i_m$, define a relation (*sd-dominance* relation) $\succeq^{\mathrm{d}}_i$ between the ith rows P_i and Q_i of P and Q, respectively, by

$$P_i \succeq^{\mathrm{d}}_i Q_i \quad \Longleftrightarrow \quad \forall \ell = 1, \cdots, m : \sum_{k=1}^{\ell} P(i, e^i_k) \geq \sum_{k=1}^{\ell} Q(i, e^i_k).$$

The random assignment Q is *sd-dominated* by P if we have $P_i \succeq^{\mathrm{d}}_i Q_i$ for all $i \in N$ and $P \neq Q$.
We say that P is *ordinally efficient* if P is not sd-dominated by any other random assignment.

"sd" stands for stochastic dominance [1].

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Let P and Q be random assignments.
For each agent $i \in N$ with preference relation $\succ_i$ given by $e_1^i \succ_i \cdots \succ_i e_m^i$, define a relation (*sd-dominance* relation) $\succeq_i^{\mathrm{d}}$ between the ith rows P_i and Q_i of P and Q, respectively, by

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Theorem 1: *The random assignment P obtained by the procedure* Random_Assignment *is ordinally efficient.*

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We say a random assignment P is *envy-free* with respect to a profile of ordinal preferences $\succ_i$ for all $i \in N$ if for all $i, j \in N$ we have $P_i \succeq_i^{\mathrm{d}} P_j$.

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We say a random assignment P is *envy-free* with respect to a profile of ordinal preferences $\succ_i$ for all $i \in N$ if for all $i, j \in N$ we have $P_i \succeq_i^{\mathrm{d}} P_j$.

Theorem 2: *The random assignment P obtained by the procedure* Random_Assignment *is envy-free.*

$\rightarrow$

Randomized Assignment

Given the random assignment P and the base x^*_P, compute a probability distribution on realizations of assignments satisfying the following:

(1) The base x^*_P is expressed as a **convex combination** of extreme bases in $\mathrm{B}(\rho)$ (characteristic vectors **of bases** $B_k \in \mathcal{B}$ $(k \in K)$):

$$x^*_P = \sum_{k \in K} \mu_k \chi_{B_k} \qquad (\mu_k > 0 \ (\forall k \in K), \ \sum_{k \in K} \mu_k = 1).$$

(2) Each $P(i, e)$ is equal to the probability that agent $i \in N$ receives good $e \in E$.

Choose an assignment according to the computed probability distribution.

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