Degree-sequences of highly-connected simple digraphs

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Current Trends in Combinatorial Optimization

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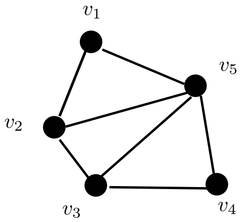
Degree sequences

Definition

The degree sequence of an undirected graph $G = (V, E)$ is a vector $d_G \in \mathbb{Z}^V$ where $d_G(v) = (\#\text{ edges in } E \text{ incident to } v)$.

Example



$$d_G = (2, 3, 3, 2, 4)$$

Question

Given $m \in \mathbb{Z}^V$, does there exist a graph G with $d_G = m$ (such that...)?

Degree sequences

Definition

The out- and in-degree sequences of an digraph $D = (V, A)$ are vectors $d_D^+, d_D^- \in \mathbb{Z}^V$ where

$$d_D^+(v) = (\#\text{ edges in } A \text{ leaving } v), \text{ and}$$
$$d_D^-(v) = (\#\text{ edges in } A \text{ entering } v).$$

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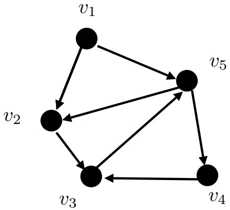

$$d_D^- = (0, 2, 2, 1, 2)$$
$$d_D^+ = (2, 1, 1, 1, 2)$$

Question

Given $m_i, m_o \in \mathbb{Z}^V$, does there exist a digraph D fitting (m_o, m_i) (that is, $d_D^+ = m_o$ and $d_D^- = m_i$) (such that...)?

Previous work

Undirected

Simple bipartite (Gale-Ryser, '57)

Simple (Erdős-Gallai, '60)

Simple k-edge-connected
(Edmonds, '64)

Simple k-node-connected
(Wang-Kleitman, '72)

(Split graphs, C_4-minor-free graphs, Unicyclic graphs, Cacti graphs, Halin graphs, . . .)

Directed

No parallel arcs (Gale-Ryser, '57)

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(Hong-Liu-Lai, '16)

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Simple degree-prescribed edge-connectivity augmentation

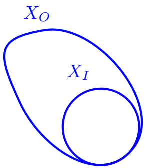
Simple k-node-connected

Bi-sets

Definition

Given a set V, a bi-set is a pair (X_O, X_I) where $X_I \subseteq X_O \subseteq V$.

- X_O is the outer, X_I is the inner member
- intersection, union
- $uv \in A$ enters (X_O, X_I) if $u \notin X_O, v \in X_I$



A digraph covers bi-set function p if $\varrho(X) \geq p(X)$ for every bi-set X.

Two bi-sets are independent if they cannot be covered by a single arc, that is, $X_I \cap Y_I = \emptyset$ or $X_O \cup Y_O = V$.

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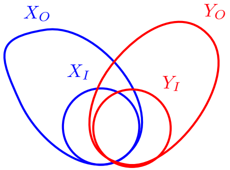

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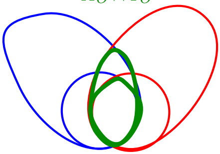

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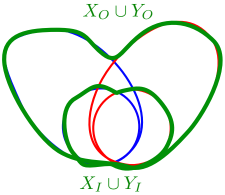

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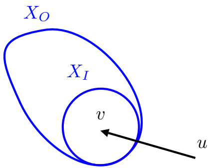

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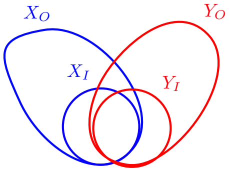

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Supermodular arc-covering

Definition

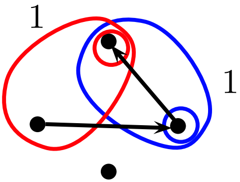
A bi-set function p is positively crossing supermodular if

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$$

whenever $p(X), p(Y) > 0$, $X_I \cap Y_I \neq \emptyset$ and $X_O \cup Y_O \neq V$.

Theorem (Frank-Jordán, '95)

The minimum number of arcs needed to cover p is equal to the maximum total p-value of an independent family of bi-sets.



Remark

When simplicity is required, the problem becomes NP-hard.

Strongly connected simple digraphs

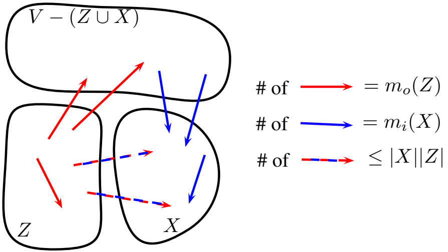
Theorem (Hong, Liu and Lai, '16)

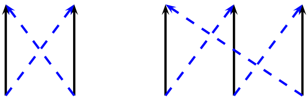
Suppose that there is a simple digraph fitting (m_o, m_i). There is a strongly connected simple digraph fitting (m_o, m_i) if and only if

$$m_o(Z) + m_i(X) - |X||Z| + 1 \leq \gamma$$

holds for every pair of disjoint subsets $X, Z \subset V$ with $X \cup Z \neq \emptyset$, where $\gamma = m_o(V) = m_i(V)$.

Necessity

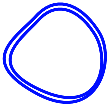
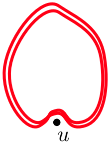
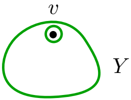




Strongly connected simple digraphs

Aim

Encode connectivity, m_o, m_i and simplicity in the same positively crossing supermodular bi-set function p.

Type 1	Type 2	Type 3
$1 < \|K\| < n-1$ 		
$p(K, K) = 1$	$p(V-u, V-u) = m_o(u)$	$p(Y, v) = m_i(v) - (\|Y\| - 1)$

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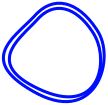
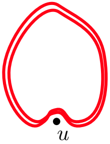
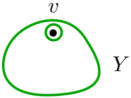
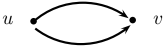
$1 < \|K\| < n-1$  $p(K,K) = 1$	 $p(V-u, V-u) = m_o(u)$	 $p(Y,v) = m_i(v) - (\|Y\|-1)$

Case 1

There is a digraph $D = (V, A)$ with γ arcs covering p.

- $\delta(v) = m_o(v)$ and $\varrho(v) = m_i(v)$ for $v \in V$
- D is simple



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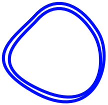
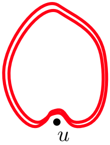
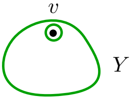
$1 < \lvert K\rvert < n-1$ 		
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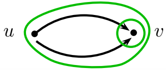

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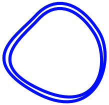	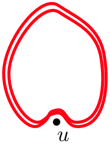	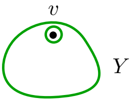
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Case 2

The minimum number of arcs needed to cover p is larger than γ.

Frank-Jordán: $\exists$ independent family $\mathcal{I}$ with $p(\mathcal{I}) > \gamma$.

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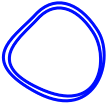
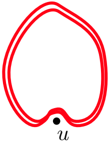
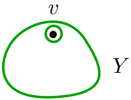
$1 < \lvert K \rvert < n-1$  $p(K,K) = 1$	 $p(V-u, V-u) = m_o(u)$	 $p(Y,v) = m_i(v) - (\lvert Y \rvert - 1)$

$\mathcal{F}$ $\qquad$ Z $\qquad$ $\mathcal{B}$

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Strongly connected simple digraphs

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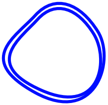
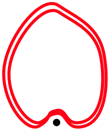
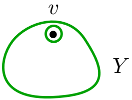
$1 < \|K\| < n - 1$  $p(K, K) = 1$	 $p(V - u, V - u) = m_o(u)$	 $p(Y, v) = m_i(v) - (\|Y\| - 1)$
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$$\wr$$

$$m_o(Z) + m_i(X) - |X||Z| > \gamma \quad \text{or} \quad m_o(Z) + m_i(X) - |X||Z| + 1 > \gamma$$

Edge-connectivity augmentation

Question

Let D_0 be a digraph. Find a simple digraph $H = (V, A)$ fitting (m_o, m_i) such that $D_0 + H$ is k-edge-connected. Conjecture?

Edge-connectivity augmentation

Question

Let D_0 be a $(k-1)$-edge-connected digraph. Find a simple digraph $H=(V,A)$ fitting (m_o, m_i) such that D_0+H is k-edge-connected.

$\mathcal{K} := \{\emptyset \neq X \subset V : \varrho_{D_0}(X) = k-1\}$ is a crossing family.

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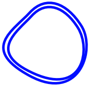
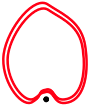
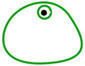
$1 < \lvert K \rvert < n-1,\ K \in \mathcal{K}$ 		
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Theorem

...if and only if there is a simple digraph fitting (m_o, m_i) and

$$m_o(Z)+m_i(X)-|X||Z|+1 \leq \gamma$$

holds for every pair of disjoint subsets $X, Z \subset V$ for which there exists $K \in \mathcal{K}$ with $Z \subseteq K \subseteq V-X$.

Further remarks

Question

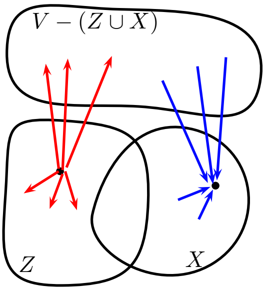
Let $D_0 = (V, A_0)$ be a $(k-1)$-edge-connected digraph and $F \subseteq A_0$. Find a digraph $H = (V, A)$ fitting (m_o, m_i) such that $D_0 + H$ is k-edge-connected and $A_0 \cap A \subseteq F$.

- Can be solved by modifying p.

An “easy” special case

$H = (V, F)$ is a digraph s.t.

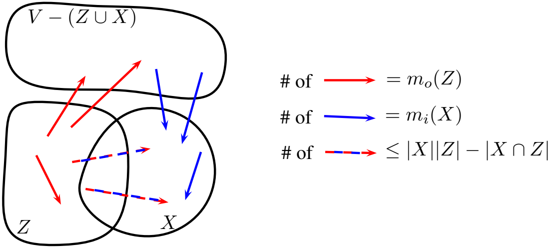
- $zv \in F$ for all $z \in Z, v \in V$,
- $ux \in F$ for all $u \in V, x \in X$.



Question

What is the minimum number of arcs needed to make H k-connected?

Finding the right condition...



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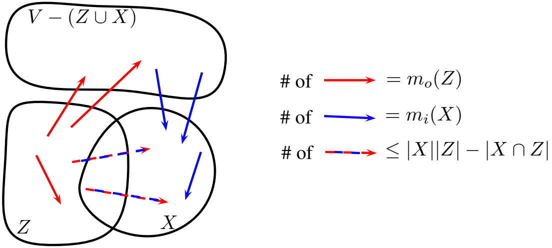
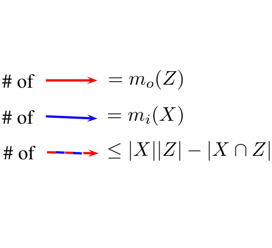


- If $\emptyset \neq Z - X \neq V$, then $\varrho(Z - X) \geq k - |X \cap Z|$.

$$\Rightarrow (m_o(Z) + m_i(X) - |X||Z| + |X \cap Z|) + (k - |X \cap Z|) \leq \gamma$$

Finding the right condition...



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Theorem

Suppose that there is a simple digraph fitting (m_o, m_i). There is a k-connected simple digraph fitting (m_o, m_i) if and only if

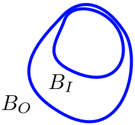
$$m_o(Z) + m_i(X) - |X||Z| + k \leq \gamma$$

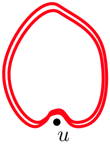
holds for distinct $X, Z \subset V$.

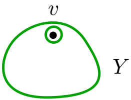
Definition of p

Lemma

$D = (V, A)$ is k-connected if and only if $\varrho(B) \geq k - |B_O - B_I|$ whenever $\emptyset \neq B_I \subseteq B_O \subset V$.

<table>
<tr><th>Type 1</th><th>Type 2</th><th>Type 3</th></tr>
<tr>
<td>

$p(B_O, B_I) =$
$k - |B_O - B_I|$</td>
<td>

$p(V - u, V - u) = m_o(u)$</td>
<td>

$p(Y, v) = m_i(v) - (|Y| - 1)$</td>
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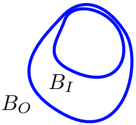
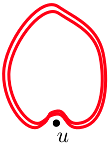
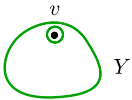
Type 1	Type 2	Type 3
 $p(B_O, B_I) = k - \lvert B_O - B_I \rvert$	 $p(V - u, V - u) = m_o(u)$	 $p(Y, v) = m_i(v) - (\lvert Y \rvert - 1)$

Case 1

p can be covered by γ arcs $\Rightarrow$ Hurray!

Case 2

Independent family $\mathcal{I}$ with $p(\mathcal{I}) > \gamma \Rightarrow$ violating pair X, Z.

Further remarks

Question

Let $D_0 = (V, A_0)$ be a digraph and $F \subseteq A_0$. Find a digraph $H = (V, A)$ fitting (m_o, m_i) such that $D_0 + H$ is k-connected and $A_0 \cap A \subseteq F$.

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- Can be solved by modifying p. ☺
- Characterization uses independent bi-sets. ☹

Open problems

Question

Characterize the degree-sequences of simple k-edge-connected digraphs.

Question

Characterize the degree-sequences of simple k-connected digraphs without using independent bi-sets.

Question

Direct algorithms...

Conjecture

The minimum number of new arcs needed to make an acyclic graph k-connected is equal to the maximum of the in- and out-degree deficiencies.