TECHNISCHE UNIVERSITÄT DARMSTADT

Mathematik

Real Complexity Theory: a numerical view on "$\mathcal{P}$ vs. $\mathcal{NP}$"

TECHNISCHE UNIVERSITÄT DARMSTADT

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Computable Real Numbers

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Theorem: For $r\in\mathbb{R}$, the following are equivalent:

Call $r\in\mathbb{R}$ computable if

There is an algorithm which, given $n\in\mathbb{N}$, prints $b_n\in\{0,1\}$ where $r=\sum_n b_n 2^{-n}$

a) r has a computable binary expansion

b) There is an algorithm printing, on input $m\in\mathbb{N}$, some $a\in\mathbb{Z}$ with $|r-a/2^{m+1}|\leq 2^{-}$[illegible]

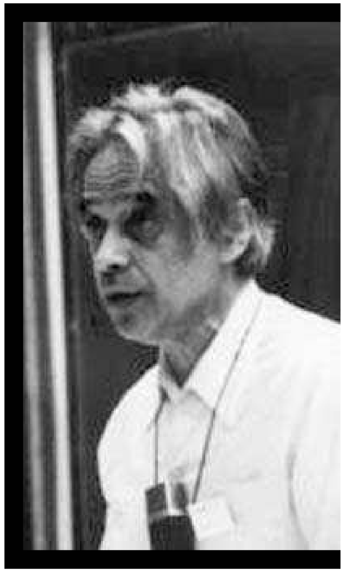
$\Leftrightarrow r\in [q_n\pm\varepsilon_n]$

c) There is an algorithm printing two sequences $(q_n)\subseteq\mathbb{Q}$ and (ε_n) with $|r-q_n|\leq\varepsilon_n\to 0$

numerators+ denominators

b) $\Leftrightarrow$ c) holds *uniformly*, $\Leftrightarrow$ a) does not [Turing'37]

interval arithmetic

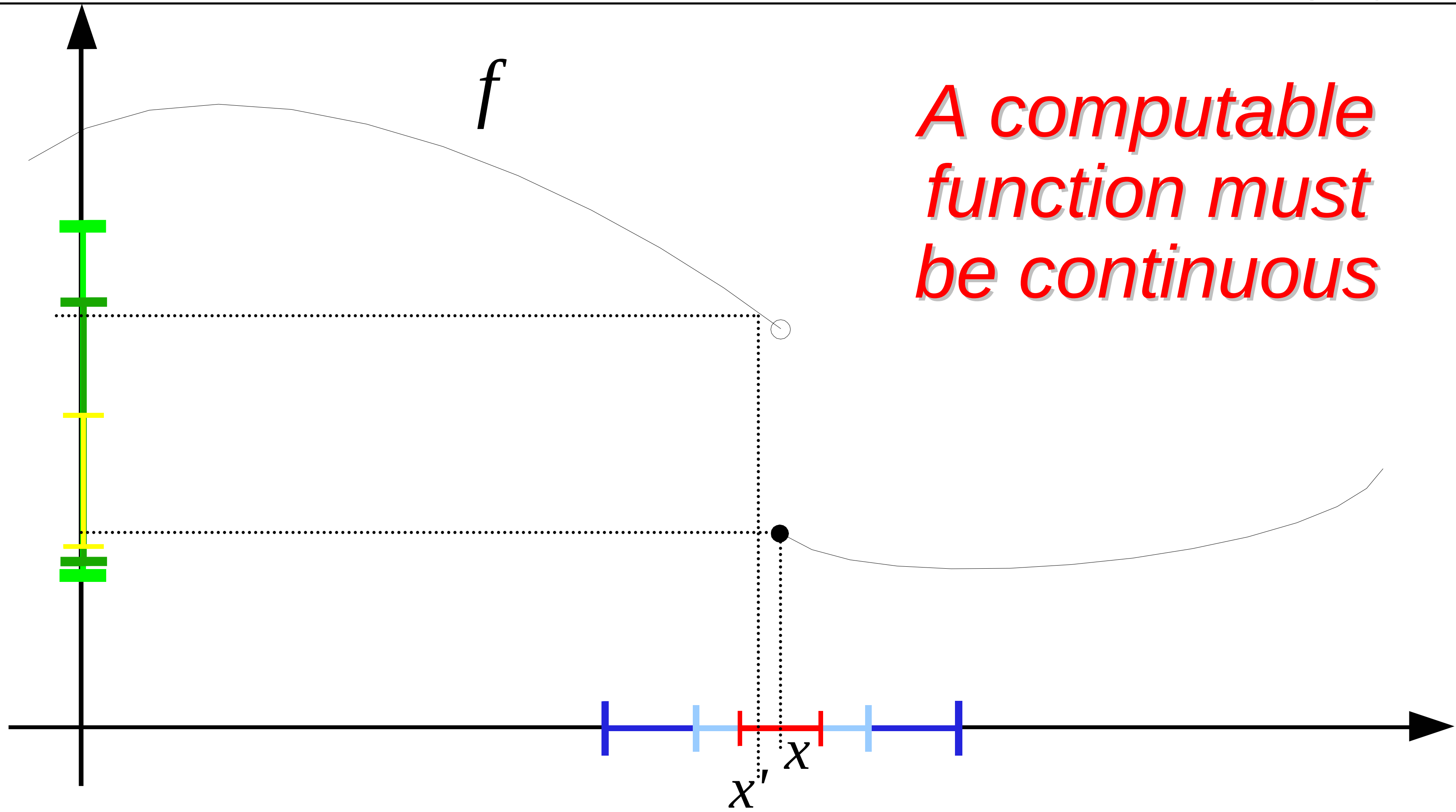


Ernst Specker (1949): (c) $\Leftrightarrow$ *Halting problem* plus (d)

d) There is an algorithm printing $(q_n)\subseteq\mathbb{Q}$ with $q_n\to r$.

$H := \{\ \langle B,\underline{x}\rangle$: algorithm B terminates on input $\underline{x}\ \}\subseteq\mathbb{N}$

Uniformly Computable Real Functions



A computable function must be continuous

$x \in \mathbb{R}$ computable $\Leftrightarrow |x - a_n/2^{n+1}| \leq 2^{-n}$ for recursive $(a_n) \subseteq \mathbb{Z}$

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Function f:[0,1]→ℝ computable **in time $t(n)$**
if some TM can, on input of $n\in\mathbb{N}$ and of
$(a_m)\subseteq\mathbb{Z}$ with $|x-a_m/2^{m+1}|\leq 2^{-m}$
in time $t(n)$ output $b\in\mathbb{Z}$ with $|f(x)-b/2^{n+1}|\leq 2^{-n}$.

iRRAM

Examples: a) $+, \times, \exp$ polytime on [0;1]!

b) $f(x)\equiv\sum_{n\in L} 4^{-n}$ iff $L\subseteq\{0,1\}^*$ polytime-decidable

c) sign, Heaviside not computable

c) 1/ln(e/x) polytime-computable

$|x-y|\leq 2^{-t(n)} \Rightarrow |f(x)-f(y)|\leq 2^{-n}$

Observation i) If f computable ⇒ continuous.
ii) If f computable in time $t(n)$, then
$t(n)$ is a modulus of uniform continuity of f.

On the Complexity of Operators

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f:[0;1]→[0;1] polytime computable (⇒ continuous)

[Friedman&Ko'82]

- Max: $f \to \mathrm{Max}(f): x \to \max\{ f(t): t \leq x \}$

  Max(f) computable in exponential time;
  polytime-computable iff $\mathcal{P}=\mathcal{NP}$

- $\int: f \to \int f: (x \to \int_0^x f(t)\, dt)$

  $\int f$ computable in exponential time;
  "#$\mathcal{P}$-complete"

even when restricting to $f \in C^\infty$

- dsolve: $C[0;1]\times[-1;1] \ni f \to z$: $\dot{z}(t)=f(t,z)$, $z(0)=0$.
  - in general no computable solution $z(t)$
  - for $f \in C^1$ $\mathcal{PSPACE}$-"complete"
  - for $f \in C^k$ $\mathcal{CH}$-"hard"

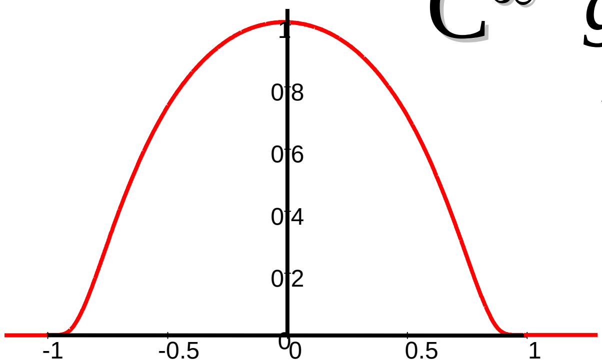
[Kawamura'10, Kawamura et al]

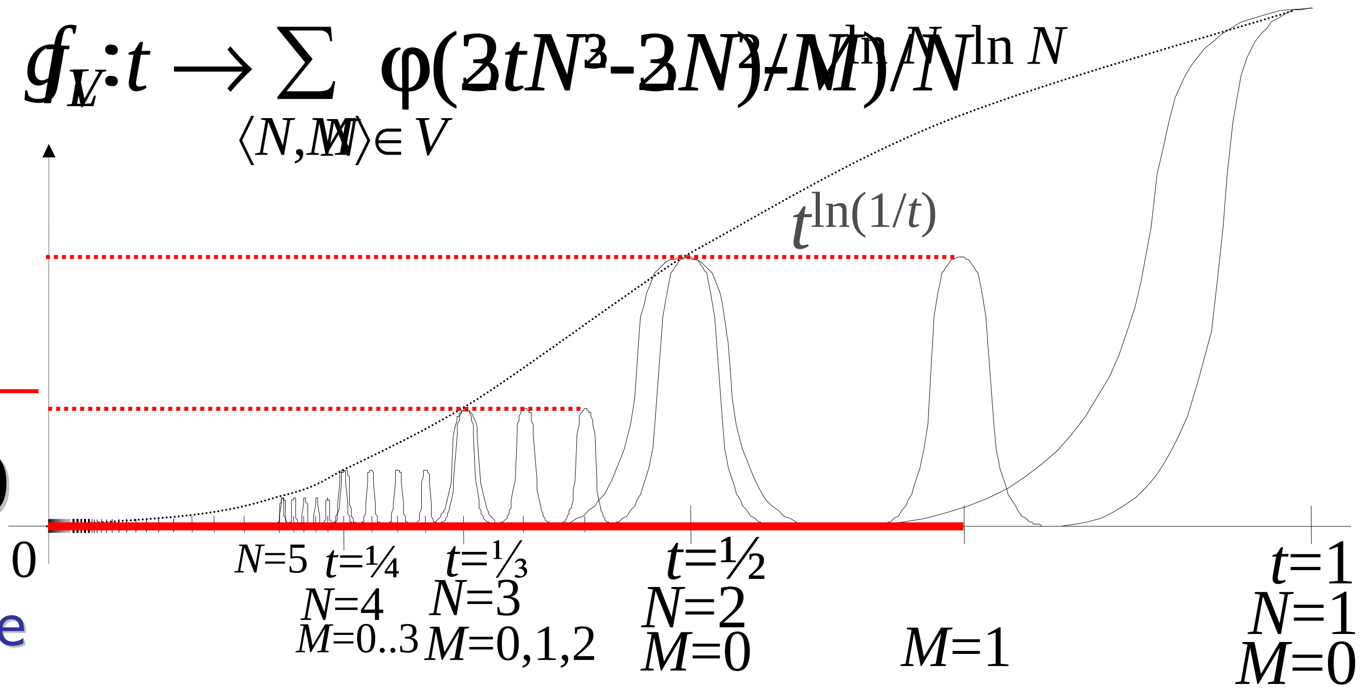
'Max is $\mathcal{NP}$-hard'

obvious in the *discrete* case

$\mathcal{NP} \ni L = \{ N\in\mathbb{N} \mid \exists M<N: \langle N,M\rangle\in V \}$, $V\in\mathcal{P}$ polytime

$$C^\infty \quad g_V: t \to \sum_{\langle N,M\rangle\in V} \varphi(2tN^2-2N-M) / N^{\ln N}$$





$\varphi(t) = \exp(-t^2/1-t^2)$

C∞ 'pulse' function
polytime computable

To every $L\in\mathcal{NP}$ there exists a polytime computable C∞ function $g_L:[0,1]\to\mathbb{R}$ s.t.: $[0,1]\ni t\to\max g_L|_{[0,t]}$ again polytime iff $L\in\mathcal{P}$

On the Complexity of Operators

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f:[0;1]→[0;1] polytime computable (⇒ continuous)

- Max: $f \to \mathrm{Max}(f): x \to \max\{\, f(t): t \leq x \,\}$ [Friedman &Ko'82]
  Max(f) computable in exponential time;
  polytime-computable iff $\mathcal{P}=\mathcal{NP}$
- $\int: f \to \int f: (x \to \int_0^x f(t)\,dt)$ non-uniform
  $\int f$ computable in exponential time;
  #$\mathcal{P}$-"complete" [Friedman&Ko'82]

even when restricting to $f \in C^\infty$

but for analytic f polytime

[N.Müller],[Bournez et al]

- dsolve: $C[0;1]\times[-1;1] \ni f \to z$: $\dot{z}(t)=f(t,z)$, $z(0)=0$.
  - in general no computable solution $z(t)$
  - for $f \in C^1$ $\mathcal{PSPACE}$-"complete" [Kawamura'10,
  - for $f \in C^k$ $\mathcal{CH}$-"hard" Kawamura et al]

Three Effects in Real Complexity

that numerical scientists might be interested in / should be aware of

cmp. "feasible real-RAM"

a) natural emergence of multivaluedness
(aka non-extensionality) $\to$ ε-semantics of "**<**"

b) Uniform computation may require *discrete advice* or otherwise 'enriched' representations (TTE)
— which yield canonical **C++** declarations

c) Parameterized uniform upper complexity bounds

Example (Brattka&Z, *Computable Spectral Thm*)
Finding an eigenvector (basis) to a given real

Example: $+$, $\exp$ computable in time polynomial in n on $[0;1]$; on $[0;2^k]$: $+$ in time polynomial in $n+k$, $\exp$ in time polynomial in $n+2^k$.

independent of x
on *compact* dom

Obstacles to a Uniform Complexity Theory of *Operators*

Evaluation $\mathrm{Eval}:(f,x)\to f(x)$

a) requires $\geq\mu(n)$ steps, $\mu:\mathbb{N}\to\mathbb{N}$ mod. of continuity to f.

"Parameter" $\mu(f)$ is not $\mathbb{N}$-valued but $\mathbb{N}^{\mathbb{N}}$-valued!

b) Even restricted to the compact domain

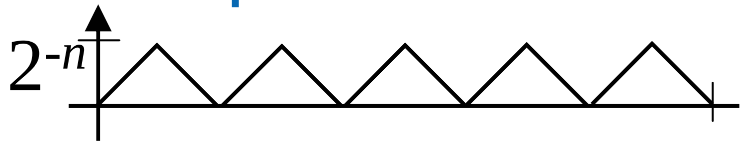
sequent. access

$\mathcal{L}_1$:= { $f:[0;1]\to[0;1]$ 1-Lipschitz } (Arzela-Ascoli)

there exists no representation $\delta:\subseteq\{0,1\}^\omega\to\mathcal{L}_1$

rendering Eval computable in *sub*exponential time.

$\geq 2^{2^{n-1}}$ functions pairwise differing when evaluating up to error 2^{-n} but only $2^{t(n)}$ different initial segments of δ-names that can be read within $t(n)$ steps. q.e.d.



$\approx 2^{n-1}$ 'hats'

Conclusion, Work in Progress, Open Questions

- Computing maxima, integrals, and ODEs is hard; and smoothness presumptions kicks in late.
- Discrete advice/enrichment and multivaluedness

o uniform complexity theory of operators
o and of higher type objects
(Calculus of Variation, Optim.Control, Shape Opt.)
o complexity of Euclidean subsets
o complexity of analytic ODEs
o complexity of PDEs
o characterize more complexity classes numerically

- Why do numerical algorithms work so well in practice? $\rightarrow$ formalization / smoothened analysis
- Transfer from theory to practice: **iRRAM**