Products of effective topological spaces and a uniformly computable Tychonoff Theorem

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The Tychonoff theorem

Topological space $\mathbf{X} = (X, \tau)$, topology τ = set of open sets

$K \subseteq X$ is compact iff every open cover of K has a finite subcover.

$$(\forall \sigma \subseteq \tau,\ K \subseteq \bigcup \sigma)\,(\exists \pi \subseteq \sigma)\,(\pi \text{ is finite and } K \subseteq \bigcup \pi)$$

Product $\mathbf{X} = (X, \tau) = \mathbf{X}_1 \times \mathbf{X}_2 \times \ldots$ of spaces $\mathbf{X}_i = (X_i, \tau_i)$:

- $X := X_1 \times X_2 \times \ldots$
- every set $X_1 \times \ldots \times X_{k-1} \times U_k \times X_{k+1} \times \ldots$ is open ($U_k \in \tau_k$),
- add all finite intersections,
- add all unions.

Theorem (Tychonoff)
The product $K_1 \times K_2 \times \ldots$ of compact sets is compact.

(accordingly for arbitrary products $\bigotimes_{i \in I} \mathbf{X}_i$ of spaces $\mathbf{X}_i$)

Computability via multi-representations

Representation approach

Computable functions on Σ^* and Σ^ω (finite and infinite sequences of symbols) are defined explicitly by Turing machines.

Computability on "abstract sets" X is defined by computations on "concrete" names from Σ^* or Σ^ω.

A representation of a set X
is a partial surjection $\delta :\subseteq Y \to X$, where $Y = \Sigma^*$ or $Y = \Sigma^\omega$.

$\delta(p) = x$ means that p is a name of x.

Examples of representations

– $\nu_{\mathbb{N}} :\subseteq \Sigma^* \to \mathbb{N}$, $\nu_{\mathbb{N}}(1111) = 4$ etc.
  addition, multiplication etc. are computable w.r.t. $\nu_{\mathbb{N}}$.

– $\rho_{\mathrm{dec}}(3.14159\ldots) = \pi$ etc. (decimal representation)

– $I :\subseteq \Sigma^* \to \mathrm{RI}$ canonical representation of $\{(a,b) \subseteq \mathbb{R} \mid a, b \in \mathbb{Q}\}$.

– $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$, $\rho(u_0\$u_1\$\ldots) = x$ iff
  $I(u_0), I(u_1), \ldots$ is a list of all rational intervals J such that $x \in J$.

There is no $(\rho, \nu_{\mathbb{N}})$-computable function $f : \mathbb{R} \to \mathbb{N}$ such that $x < f(x)$, but:

There is a computable function $h : \Sigma^\omega \to \Sigma^*$ such that

$$\rho(p) = x \implies x < \nu_{\mathbb{N}} \circ h(p) \ .$$

The multi-function $f : \mathbb{R} \rightrightarrows \mathbb{N}$ is $(\rho, \nu_{\mathbb{N}})$-computable, where

$f(x)$ is some $n \in \mathbb{N}$ such that $x < n$.

Computable Analysis needs multi-functions.

Computable Analysis needs multi-representations.

A multi-representation of a set X is a
surjective multi-function $\delta : Y \rightrightarrows X$, where $Y = \Sigma^*$ or $Y = \Sigma^\omega$.

$x \in \delta(p)$ means that p is a name of x,
p may be the name of many $x \in X$.
(Many people have the name Miller.)

Induced computability

Multi-representations $\gamma : \Sigma^\omega \rightrightarrows X$ and $\gamma_0 : \Sigma^\omega \rightrightarrows X_0$
$f :\subseteq \Sigma^\omega \to \Sigma^\omega$ operates on "concrete data",
$g : X_1 \rightrightarrows X_0$ operates on "abstract data".

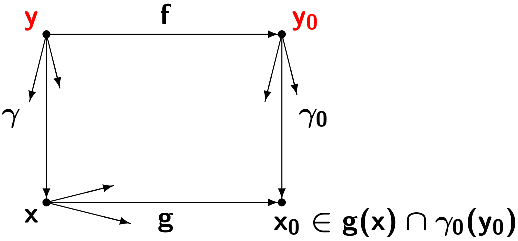

Figure : f realizes g via (γ, γ_0).

g is computable iff it has a computable realization.

Tychonoff on computable topological spaces

Definition effective topological space $\mathbf{X} = (X, \tau, \beta, \nu)$
- τ is T_0: $x = y$ if $\{U \in \tau \mid x \in U\} = \{U \in \tau \mid y \in U\}$
- $\beta \subseteq \tau$ is a countable base
- $\nu :\subseteq \Sigma^* \to \beta$ is a representation of β

Definition $\mathbf{X} = (X, \tau, \beta, \nu)$ is computable iff
- $\mathrm{dom}(\nu) \in \Sigma^*$ is recursive
- There is some r.e. set $S \subseteq (\mathrm{dom}(\alpha))^3$ such that

$$\nu(u) \cap \nu(v) = \bigcup \{\nu(w) \mid (u, v, w) \in S\}$$

Examples: $(\mathbb{R}, \tau_{\mathbb{R}}, \mathrm{RI}, I)$, $\mathbb{R}^2$ with rational open Euclidean balls

Definition product of $\mathbf{X_1} = (X_1, \tau_1, \beta_1, \nu_1)$, $\mathbf{X_2} = (X_2, \tau_2, \beta_2, \nu_2)$:

$\mathbf{X_1} \times \mathbf{X}_2 := (X_1 \times X_2, \tau, \beta, \nu)$, $\nu(u_1 \$ u_2) := \nu_1(u_1) \times \nu_2(u_2)$

Theorem $\mathbf{X_1} \times \mathbf{X}_2$ is computable if $\mathbf{X}_1$ and $\mathbf{X}_2$ are computable.

for an effective topological space $\mathbf{X} = (X, \tau, \beta, \nu)$:

Definition (Multi-representation κ_{X} of compact sets)

$K \in \kappa_{\mathbf{X}}(p)$ iff p is (encodes)
a list of all $\{u_1, \ldots, u_k\}$ such that $K \subseteq \nu(u_1) \cup \ldots \cup \nu(u_k)$.

Theorem (Tychonoff I)
For computable topological spaces $\mathbf{X_1}$ and $\mathbf{X_2}$ with product $\mathbf{X} = \mathbf{X_1} \times \mathbf{X_2}$, the product operator on compact sets,

$$(K_1, K_2) \mapsto K_1 \times K_2, \quad \text{is} \quad (\kappa_{\mathbf{X}_1}, \kappa_{\mathbf{X}_2}, \kappa_{\mathbf{X}})\text{-computable}.$$

Corollary
The product of computable compact sets is computable compact.

Computable Tychonoff on the class of effectiv topological spaces

Definition Products of eff. top. spaces $\mathbf{X}_i = (X_i, \tau_i, \beta_i, \nu_i)$

$$\mathbf{X}_1 \times \mathbf{X}_2 := (X_1 \times X_2, \tau, \beta, \nu), \quad \nu(u_1 \$ u_2) := \nu_1(u_1) \times \nu_2(u_2)$$

$$\mathbf{X}_1 \times \ldots \times \mathbf{X}_n \times \mathbf{X}_{n+1} := (\mathbf{X}_1 \times \ldots \times \mathbf{X}_n) \times \mathbf{X}_{n+1}$$

$\mathbf{X}_1 \times \mathbf{X}_2 \times \ldots := \mathbf{Y} := (Y, \tau_{\mathbf{Y}}, \beta_{\mathbf{Y}}, \nu_{\mathbf{Y}})$ where

$$\begin{aligned} Y &:= X_1 \times X_2 \times \ldots, \\ \nu_{\mathbf{Y}}\langle u_1, \ldots, u_k \rangle &:= \nu_1(u_1) \times \nu_2(u_2) \times \ldots \times \nu_k(u_k) \times X_{k+1} \times X_{k+2} \times \ldots \\ \tau_{\mathbf{Y}} &:= \text{the topology on } \mathbf{Y} \text{ generated by } \mathrm{range}(\nu_{\mathbf{Y}}). \end{aligned}$$

Multi-representation of all eff. top. spaces

The concrete data of an eff. top. space $\mathbf{X} = (X, \tau, \beta, \nu)$:
– the characteristic function of $\mathrm{dom}(\nu)$ and
– an enumeration of some set S such that

$$\nu(u) \cap \nu(v) = \bigcup\{\nu(w) \mid (u, v, w \in S\}.$$

Definition (Multi-representation $\Delta : \Sigma^\omega \rightrightarrows \mathcal{T}$ of the class $\mathcal{T}$ of all effective topological spaces)

$\mathbf{X} = (X, \tau, \beta, \nu) \in \Delta\langle r, s\rangle$ iff
– r enumerates the graph of the characteristic function of $\mathrm{dom}(\nu)$
– s enumerates a subset $S \subseteq (\mathrm{dom}(\nu))^3$ such that

$$\nu(u) \cap \nu(v) = \bigcup\{\nu(w) \mid (u, v, w \in S\}$$

Remark Spaces with the same name may be not homeomorphic.
$\Delta(p)$ has a maximal element $\mathbf{X}_p$.

Definition For multi-representations $\delta_i : \Sigma^\omega \rightrightarrows Z_i$:

$$[\delta_1, \delta_2, \ldots]^+(1^n 0\langle p_1, \ldots, p_n\rangle) := \{n\} \times \delta_1(p_1) \times \ldots \times \delta_n(p_n)\,,$$

$$[\delta_1, \delta_2, \ldots]\langle p_1, p_2, \ldots\rangle := \delta_1(p_1) \times \delta_2(p_2) \times \ldots$$

Theorem (products of spaces)

$(\mathbf{X}_1, \mathbf{X}_2) \mapsto \mathbf{X}_1 \times \mathbf{X}_2$
is (Δ, Δ, Δ)-computable

$(n, \mathbf{X}_1, \mathbf{X}_2, \ldots) \mapsto \mathbf{X}_1 \times \mathbf{X}_2, \times \ldots \times \mathbf{X}_n$
is $([\Delta, \Delta, \ldots]^+, \Delta)$-computable,

$(\mathbf{X}_1, \mathbf{X}_2, \ldots) \mapsto \mathbf{X}_1 \times \mathbf{X}_2, \times \ldots$
is $([\Delta, \Delta, \ldots], \Delta)$-computable.

Definition
Multi-representation of the class of all compact subsets of effective topological spaces:

$$(\mathbf{X}, K) \in \kappa^{\Delta}\langle p, q\rangle \iff \mathbf{X} \in \Delta(p) \text{ and } K \in \kappa_{\mathbf{X}}(q)$$

Theorem (computable Tychonoff)

$$((\mathbf{X}_1, K_1), (\mathbf{X}_2, K_2)) \mapsto (\mathbf{X}_1 \times \mathbf{X}_2, K_1 \times K_2)$$

is $(\kappa^{\Delta}, \kappa^{\Delta}, \kappa^{\Delta})$-computable.

$$(n, (\mathbf{X}_1, K_1), (\mathbf{X}_2, K_2), \ldots) \mapsto (\mathbf{X}_1 \times \ldots \times \mathbf{X}_n, K_1 \times \ldots \times K_n)$$

is $([\kappa^{\Delta}, \kappa^{\Delta}, \ldots]^{+}, \kappa^{\Delta})$-computable.

$$((\mathbf{X}_1, K_1), (\mathbf{X}_2, K_2), \ldots) \mapsto (\mathbf{X}_1 \times \mathbf{X}_2 \times \ldots, K_1 \times K_2 \times \ldots)$$

is $([\kappa^{\Delta}, \kappa^{\Delta}, \ldots], \kappa^{\Delta})$-computable.

Corollaries . . .

Products of sets and Tychonoff for mincover

Definition Mincover representation of compact sets

– For $\mathbf{X} = (X, \tau, \beta, \nu)$, and compact $K \subseteq X$,

$K \in \widetilde{\kappa}_{\mathbf{X}}(p)$ iff p is (encodes) a list of all sets $\{u_1, \ldots, u_k\}$ such that $K \subseteq \nu(u_1) \cup \ldots \cup \nu(u_k)\}$ and $(\forall\, i)\, K \cap \nu(u_i) \neq \emptyset$.

– $(\mathbf{X}, K) \in \widetilde{\kappa}^{\Delta}\langle p, q\rangle$ iff $\mathbf{X} \in \delta(p)$ and $K \in \widetilde{\kappa}_{\mathbf{X}}(q)$.

Computable Tychonoff also for mincover?

For effective topological space $\mathbf{X} = (X, \tau, \beta, \nu)$

Definition (multi-representation of 2^X:)
$A \in \widetilde{\psi}_{\mathbf{X}}(p) \iff p$ is a list of all u such that $A \cap \nu(u) \neq \emptyset$.

Remark for open U:

$$A \cap U \neq \emptyset \iff \overline{A} \cap U \neq \emptyset$$
$$A \in \widetilde{\psi}_{\mathbf{X}}(p) \iff \overline{A} \in \widetilde{\psi}_{\mathbf{X}}(p)$$

$\widetilde{\psi}_{\mathbf{X}}$ generalizes the positive representation ψ^+ of the set of closed sets.

There is a computable T_1-space with non-closed compact sets.

Definition
Multi-representation of the class of all subsets of all effective topological spaces:

$$(\mathbf{X}, A) \in \widetilde{\psi}^{\Delta}\langle p, q\rangle \iff \mathbf{X} \in \Delta(p) \quad \text{and} \quad A \in \widetilde{\psi}_{\mathbf{X}}(q)$$

Theorem (computable Cartesian products)

$$((\mathbf{X}_1, A_1), (\mathbf{X}_2, A_2)) \mapsto (\mathbf{X}_1 \times \mathbf{X}_2, A_1 \times A_2)$$

is $(\widetilde{\psi}^{\Delta}, \widetilde{\psi}^{\Delta}, \widetilde{\psi}^{\Delta})$-computable.

$$(n, (\mathbf{X}_1, A_1), (\mathbf{X}_2, A_2), \ldots) \mapsto (\mathbf{X}_1 \times \ldots \times \mathbf{X}_n, A_1 \times \ldots \times A_n)$$

is $([\widetilde{\psi}^{\Delta}, \widetilde{\psi}^{\Delta}, \ldots]^{+}, \widetilde{\psi}^{\Delta})$-computable.

$$((\mathbf{X}_1, A_1), (\mathbf{X}_2, A_2), \ldots) \mapsto (\mathbf{X}_1 \times \mathbf{X}_2 \times \ldots, A_1 \times A_2 \times \ldots)$$

is $([\widetilde{\psi}^{\Delta}, \widetilde{\psi}^{\Delta}, \ldots], \widetilde{\psi}^{\Delta})$-computable.

Corollaries . . .

Lemma $\widetilde{\kappa}^\Delta \equiv \kappa^\Delta \wedge \widetilde{\psi}^\Delta$

where $(\kappa^\Delta \wedge \widetilde{\psi}^\Delta)\langle p, q\rangle := \kappa^\Delta(p) \cap \widetilde{\psi}^\Delta(q)$

By this lemma and the above theorems for κ^Δ and $\widetilde{\psi}^\Delta$:

Theorem (computable Tychonoff for mincover)

$$((\mathbf{X}_1, K_1), (\mathbf{X}_2, K_2)) \mapsto (\mathbf{X}_1 \times \mathbf{X}_2, K_1 \times K_2)$$

is $(\widetilde{\kappa}^\Delta, \widetilde{\kappa}^\Delta, \widetilde{\kappa}^\Delta)$-computable.

$$(n, (\mathbf{X}_1, K_1), (\mathbf{X}_2, K_2), \ldots) \mapsto (\mathbf{X}_1 \times \ldots \times \mathbf{X}_n, K_1 \times \ldots \times K_n)$$

is $([\widetilde{\kappa}^\Delta, \widetilde{\kappa}^\Delta, \ldots]^+, \widetilde{\kappa}^\Delta)$-computable.

$$((\mathbf{X}_1, K_1), (\mathbf{X}_2, K_2), \ldots) \mapsto (\mathbf{X}_1 \times \mathbf{X}_2 \times \ldots, K_1 \times K_2 \times \ldots)$$

is $([\widetilde{\kappa}^\Delta, \widetilde{\kappa}^\Delta, \ldots], \widetilde{\kappa}^\Delta)$-computable.

Corollaries . . .

Accordingly multi-representations of other classes of structures:

effective Banach spaces, effective metric spaces, effective Hilbert spaces, effective measure spaces, etc.

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