The Continuity Problem, Once Again

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Work in progress within the EU-Project COMPUTAL

Shonan Meeting on ICC and Applications
Shonan Village Center, 4-7 November 2013

Let $\mathcal{T} = (T, \tau)$ be a topological space with countable basis

$$B_0, B_1, \ldots, B_n, \ldots$$

Since we are interested in effectivity **morphisms** $F\colon \mathcal{T} \to \mathcal{T}'$ of the respective category are

effectively continuous maps,

i.e., for each $n \geq 0$,

$F^{-1}(B'_n)$ is an effective union of basic open sets, uniformly in n.

We are working in the framework of Recursive Mathematics, i.e. all elements of T are computable. By coding the respective programs a (partial) indexing (numbering)

$$x\colon \omega \rightharpoonup T \text{ onto}$$

of T is obtained.

Then (T, x) is a numbered set. **Morphisms** $F\colon T \to T'$ of the respective category are

effective maps,

i.e. maps *tracked* by some partial computable function f on the code, which means that for $i \in \mathrm{dom}(x)$,

$$F(x_i) = x'_{f(i)}.$$

The relationship between both effectivity notions is known as the *continuity problem*.

Mostly, one easily shows that effectively continuous maps are effective. The converse implication is hard, in general.

Early results:

Myhill/Shepherson (1955)

Operations on the partial computable functions are effective if, and only if, they are effectively continuous.

Kreisel/Lacombe/Shoenfield (1959)

Operations on the total computable functions are effective if, and only if, they are effectively continuous.

Egli/Constable, Smyth, Weihrauch, ... (1976, 1977, 1980)

Functions on the computable points of an effectively given domain are effective if, and only if, they are effectively continuous.

Ceĭtin, Moschovakis (1962, 1964)

Functions on recursive metric spaces are effective if, and only if, they are effectively continuous.

General case (Spreen/Young, Spreen (1984, 1998)):

Functions $F \colon \mathcal{T} \to \mathcal{T}'$ are effectively continuous if, and only if, they are effective and have a witness for noninclusion,
i.e.,

$$x_i \in F^{-1}(B'_n), \quad B_m \not\subseteq F^{-1}(B'_n) \qquad \Longrightarrow$$

a witness $z \in B_m \setminus F^{-1}(B'_n)$ can effectively be found, uniformly in i, m, n.

Note.

Domain case Effective functions do have a witness for noninclusion.

Metric case Effective functions do have a witness for noninclusion.

Not true in general!

In this talk we will report on the following observation:

- F effective $\Longrightarrow$ F effectively sequentially continuous.
- F effectively sequentially continuous *and* F has witness for noninclusion

$$\Longrightarrow \quad F \text{ effectively continuous.}$$

As is known, in general

$$\text{sequential continuity} \quad \not\Longrightarrow \quad \text{continuity.}$$

The implication holds for spaces with countable basis which is true in our case. However,

$$\text{effective sequential continuity} \quad \not\Longrightarrow \quad \text{effective continuity.}$$

The witness property is needed in addition!

Definition

A sequence $(y_a)_a$ of points of T is *computable* if there is a total computable function f such that for all $a \geq 0$,

$$y_a = x_{f(a)}.$$

Assumptions

- Let SEQ be a set of computable sequences of points of T so that
  - $(y_a)_a \in \mathrm{SEQ} \quad \Longrightarrow (\forall c \geq 0) \begin{pmatrix} y_a & a < c \\ y_c & a \geq c \end{pmatrix}_a \in \mathrm{SEQ}$
  - $(y_a)_a \in \mathrm{SEQ} \quad \Longrightarrow \quad (\forall c \geq 0)(\forall N \geq c)$
    $$\begin{pmatrix} y_a & a < c \\ y_c & c \leq a \leq N \\ y_{c+(a-N)} & a > N \end{pmatrix}_a \in \mathrm{SEQ}.$$

- The set $\lim y_a$ of all limit points of $(y_a)_a$ has a greatest element $\mathrm{LIM} y_a$ with respect to the specialization order defined by
$$z_1 \leq_\tau z_2 \iff (\forall n \geq 0)[z_1 \in B_n \Rightarrow z_2 \in B_n].$$
- Let $L \subseteq \omega$ be computably enumerable so that for all $i \in \mathrm{dom}(x)$ and $n \in \omega$,
$$x_i \in B_n \iff \langle i, n \rangle \in L.$$
- There is a partial computable function li such that for $m \geq 0$,
$$(x_{\varphi_m(a)})_a \text{ converges } \implies \mathrm{li}(m)\downarrow \in \mathrm{dom}(x) \wedge x_{\mathrm{li}(m)} = \mathrm{LIM} x_{\varphi_m(a)}.$$
Here φ is a Gödel numbering of the partial computable functions.

Step 1 Every effective map F is effectively sequentially continuous.

Let f track F.

Given $(x_{\varphi_m(a)})_a$ convergent with $F(\mathrm{LIM} x_{\varphi_m(a)}) \in B'_n$, we use the recursion theorem to construct a computable sequence

$$(z_a)_a = (x_{\varphi_{h(m,n)}(a)})_a$$

as follows:

1. Follow the sequence $(x_{\varphi_m(a)})_a$ as long as

$$F(\mathrm{LIM} z_a)$$

has **not** been found in B'_n, or $(z_a)_a$ does not converge.

2. If $(z_a)_a$ converges and $F(\mathrm{LIM} z_a)$ has been found in B'_n, say in step N_0, repeat $x_{\varphi_m(N_0)}$ as long as

$$F(x_{\varphi_m(N_0)})$$

has **not** been found in B'_n.

3. If, in step N_1, $F(x_{\varphi_m(N_0)})$ has been found in B'_n, repeat $x_{\varphi_m(N_0+1)}$ as long as

$$F(x_{\varphi_m(N_0+1)})$$

has **not** been found in B'_n.

4. ...

Suppose 1: $(z_a)_a$ does not converge, or $F(\mathrm{LIM} z_a)$ will never be found in B'_n.

Then

$$z_a = x_{\varphi_m(a)},$$

for all $a \geq 0$. Hence, $(z_a)_a$ converges with $\mathrm{LIM} z_a = \mathrm{LIM} x_{\varphi_m(a)}$. But $F(\mathrm{LIM} x_{\varphi_m(a)}) \in B'_n$, by assumption, a contradiction.

Thus N_0 exists and depends computably on m, n.

Suppose 2: $F(x_{\varphi_m(N_0)})$ will never be found in B'_n.

Then

$$\mathrm{LIM} z_a = x_{\varphi_m(N_0)}.$$

As we have just seen, $F(\mathrm{LIM} z_a)$ will be found in B'_n, i.e. $F(x_{\varphi_m(N_0)})$ will be found in B'_n, contradiction.

Thus, N_1 exists and depends computably on m, n.

By induction it follows that for all $c \geq N_0$,

$$F(x_{\varphi_m(c)}) \in B'_n.$$

Thus, $(F(x_{\varphi_m(a)}))_a$ converges to $F(\mathrm{LIM} x_{\varphi_m(a)})$.

Step 2 Let F have a witness for noninclusion. Then, if F is effectively sequentially continuous, it is effectively pointwise continuous.

Definition

F is *effectively pointwise continuous* if there is a two-place partial computable function h such that for all $i \in \mathrm{dom}(x)$ and $n \geq 0$ with $F(x_i) \in B'_n$,

$$h(i,n){\downarrow} \wedge x_i \in B_{h(i,n)} \subseteq F^{-1}(B'_n).$$

Definition

Let $\prec \subseteq \omega^2$ be transitive. Then $\prec$ is a *strong inclusion* if for all $m, n \in \omega$,

$$m \prec n \Longrightarrow B_m \subseteq B_n.$$

Assumption

- $\{ B_n \mid n \geq 0 \}$ is a *strong basis*, i.e.,

$$(\forall z \in T)(\forall m, n \geq 0)[z \in B_m \cap B_n \Rightarrow (\exists a \prec m, n) z \in B_a.$$

- $\prec$ is computably enumerable

Definition

Let φ_m be decreasing with respect to $\prec$. Then $(B_{\varphi_m(a)})_a$ *converges* to $y \in T$, if $\{ B_a \mid a \in \mathrm{range}(\varphi_m) \}$ is a strong base of the neighbourhood filter $\mathcal{N}(y)$

Lemma

There is a total computable function q such that for all $i \in \mathrm{dom}(x)$, $(B_{\varphi_{q(i)}(a)})_a$ converges to x_i.

Assumption

If, for all $a \geq 0$, $y_a \in B_{\varphi_{q(i)}(a)}$, then $(y_a)_a \in \mathrm{SEQ}$.

Suppose: $F(x_i) \in B'_n$, but $F(B_{\varphi_{q(i)}(a)}) \not\subseteq B'_n$, for all $a \geq 0$.

Since F has a witness for noninclusion, a computable sequence $(y_a)_a$ can be constructed, uniformly in i, n such that for $a \geq 0$,

$$y_a \in B_{\varphi_{q(i)}(a)} \setminus F^{-1}(B'_n).$$

As $\{ B_c \mid c \in \mathrm{range}(\varphi_{q(i)}) \}$ is a strong base of $\mathcal{N}(x_i)$, $\mathrm{LIM} y_a = x_i$. Hence,

$$F(\mathrm{LIM} y_a) \in B'_n.$$

By Step 1 it follows that

$$F(y_{N_0}) \in B'_n,$$

where N_0 computably depends on i, n. Contradiction!

Thus,

$$F(B_{\varphi_{q(i)}(N_0(i,n))}) \subseteq B'_n,$$

i.e., F is effectively pointwise continuous.

Step 3 Every effectively pointwise continuous map is effectively continuous.

Holds if $\mathcal{T}$ is *recursively separable*, i.e. contains an effectively enumerable dense subset.

Future work. Extension to effectively given limit spaces. Morphisms of the respective category are effectively sequentially continuous maps.