About Representations of and Operators on Subsets of $\mathbb{R}^d$

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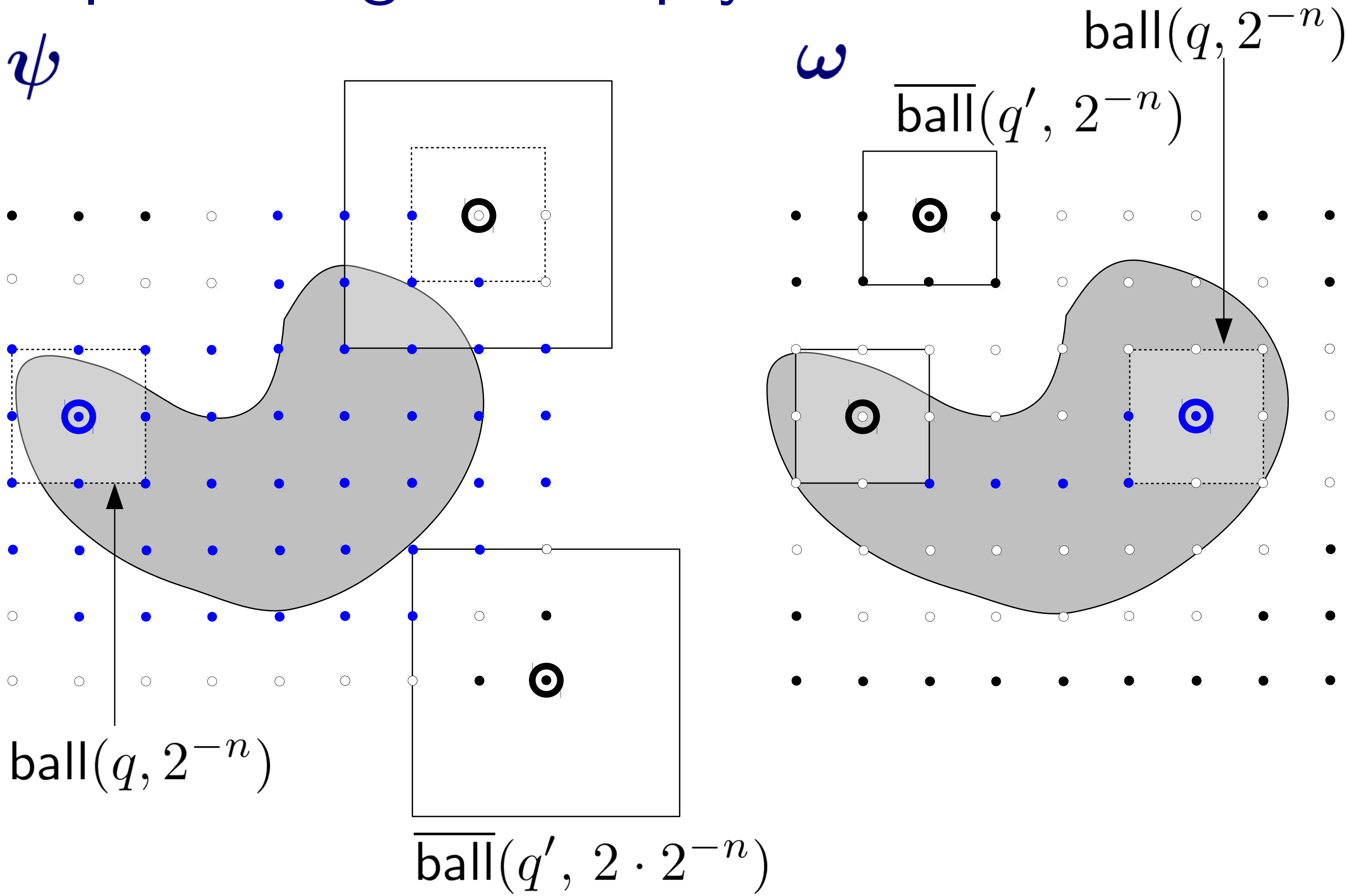
Quick recap of 2nd-order representations

- Representation of X: $\boldsymbol{\xi} :\subseteq \left(\Sigma^* \to \Sigma^*\right) \twoheadrightarrow X$
- Uniform computation of operators
- 2nd-order polynomial bounds
- Enrichments: produces new representations

Goal: explicitly stating crucial problem-specific parameters and their influence on the complexity

Part I: Comparing ψ with ω

Representing non-empty closed sets



Comparing ψ and ω

Fact [Ziegler'02, Hertling'02]

Both representations are computably equivalent (for the class of closed regular sets).

Theorem (heavily relying on Grötschel/Lovász/Schrijver'88)

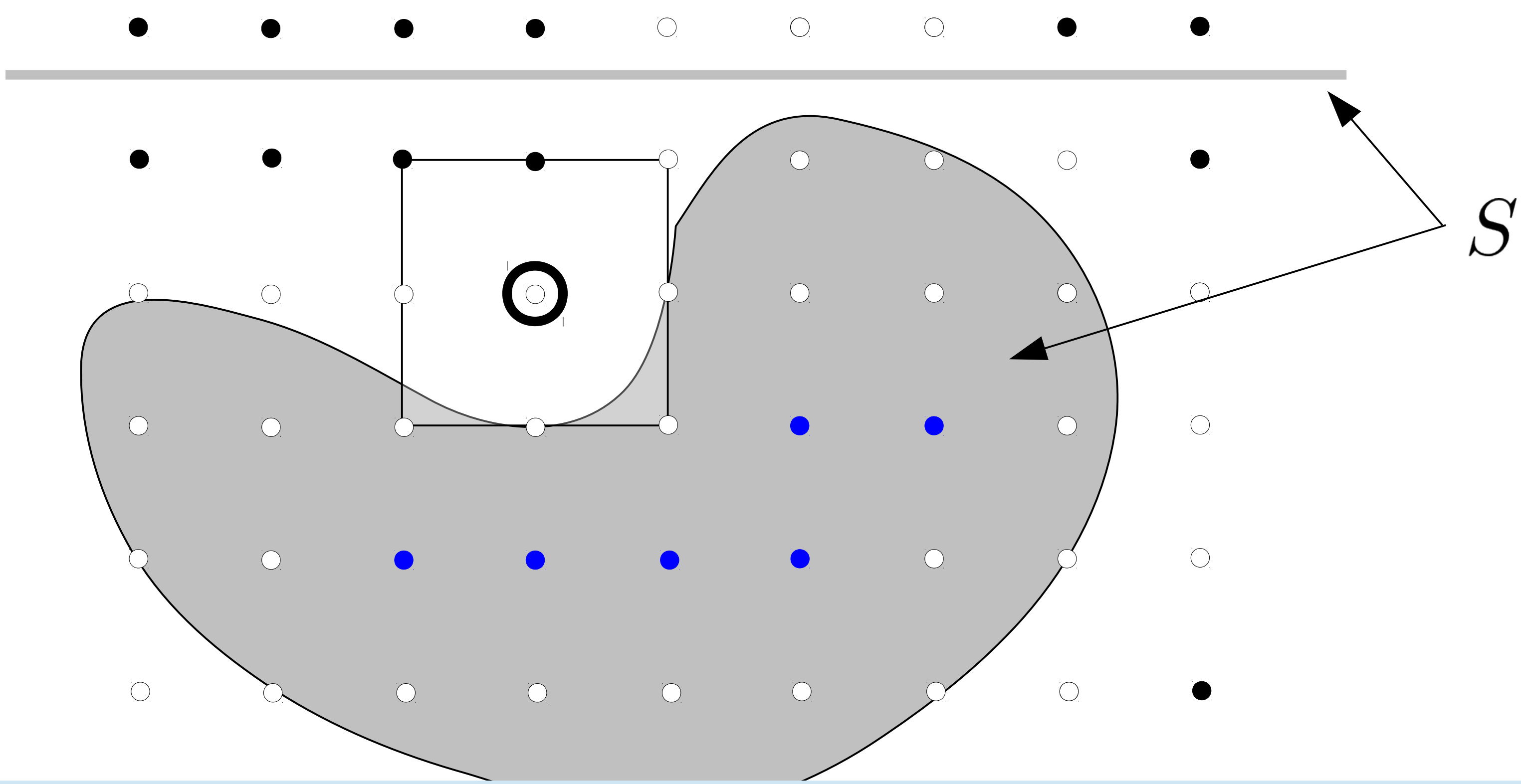
Using enrichments, they even become second-order polynomial-time equivalent:

$$\psi \preceq_{\mathrm{p}} \omega$$

$$\left[\omega \sqcap \text{some parameters}\right] \preceq_{\mathrm{p}} \psi$$

Almost the correct statement :)

Challenging part: $\boldsymbol{\omega} \sqcap (\ldots) \preceq_{\mathsf{p}} \boldsymbol{\psi}$



Restrict to $\mathcal{R} := \{S \in \mathcal{A} \mid \overline{S^\circ} = S\}$

$$\mathcal{K} := \{S \in \mathcal{A} \mid S \text{ is compact}\}$$

$$\mathcal{C} := \{S \in \mathcal{A} \mid S \text{ is convex}\}$$

Proof sketch

Theorem (heavily relying on Grötschel/Lovász/Schrijver'88)

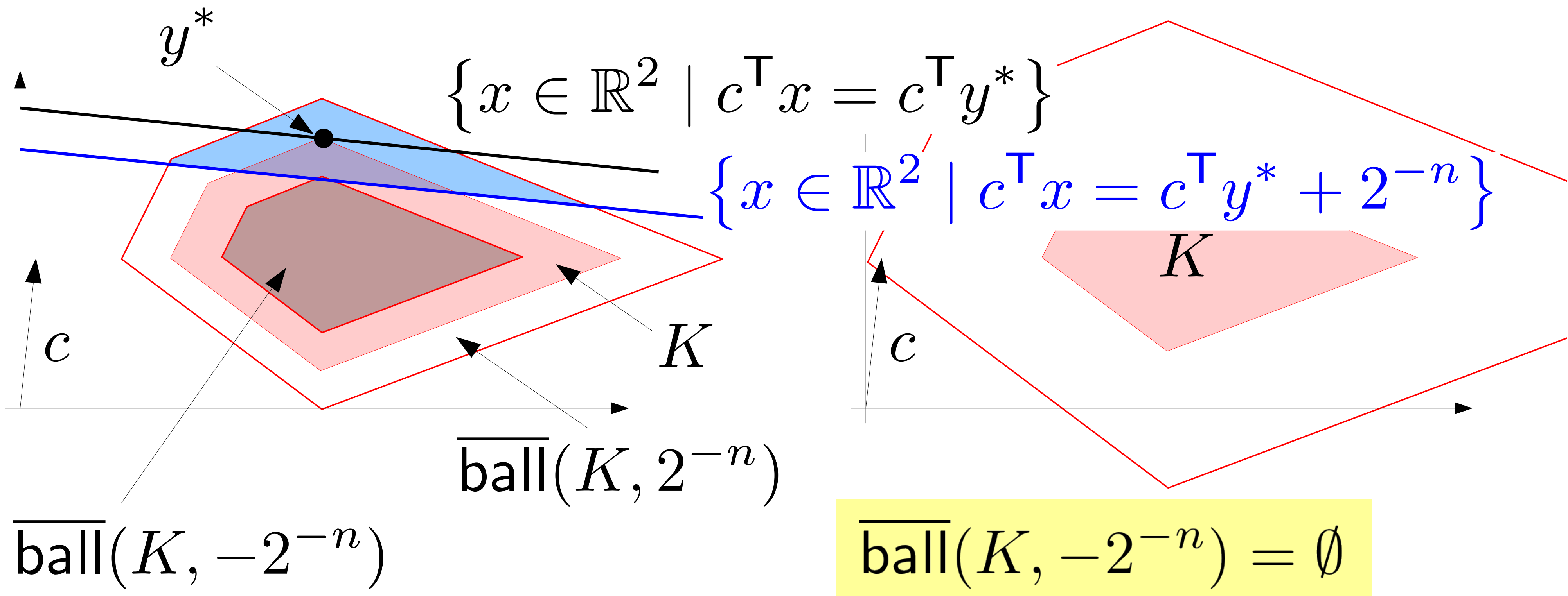
$$\left[\boldsymbol{\omega}|^{\mathcal{KCR}} \sqcap \mathsf{bin}(a) \sqcap \mathsf{un}(r) \sqcap \mathsf{un}(b)\right] \preceq_{\mathsf{p}} \boldsymbol{\psi}|^{\mathcal{KCR}}$$

1. Define "Weak Optimization Problem" (WOPT)
2. Reduce $\boldsymbol{\omega}|^{\mathcal{KCR}} \sqcap \mathsf{bin}(a) \sqcap \mathsf{un}(r) \sqcap \mathsf{un}(b)$ to WOPT
3. Define polar $K^\bullet$ of a set K
4. Reduce WOPT to $\boldsymbol{\psi}$

Proof sketch

1. Definition of WOPT

Given: $c \in \mathbb{R}^d$ with $\|c\| = 1$, and $n \in \mathbb{N}$.

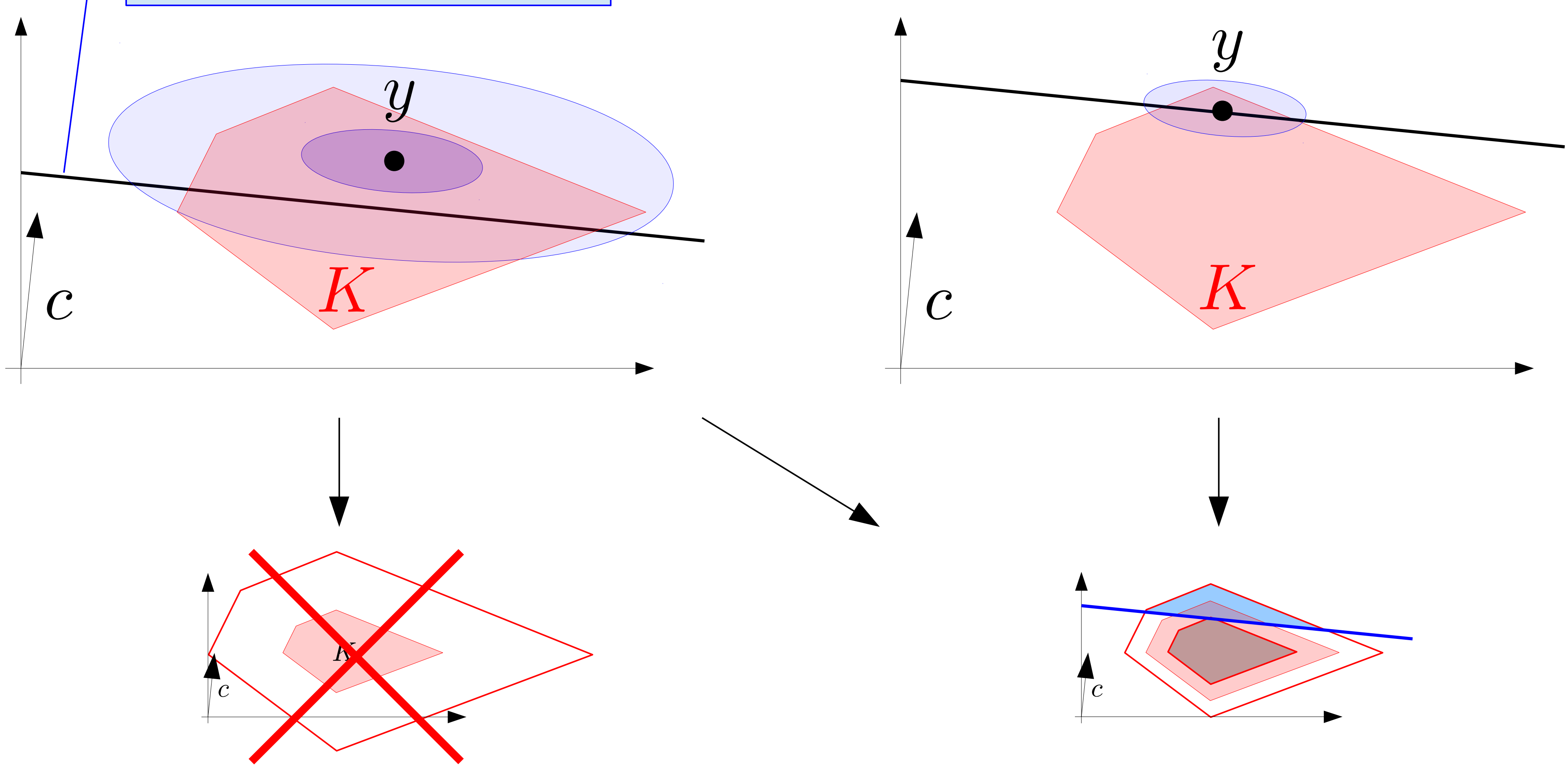


Proof sketch

2. Reduce $\omega|^{\mathcal{KCR}} \sqcap \mathrm{bin}(a) \sqcap \mathrm{un}(r) \sqcap \mathrm{un}(b)$ to WOPT

Given: ω-name, parameters a, r, b, normalized $c \in \mathbb{R}^d$, $\gamma \in \mathbb{D}$, and $n \in \mathbb{N}$

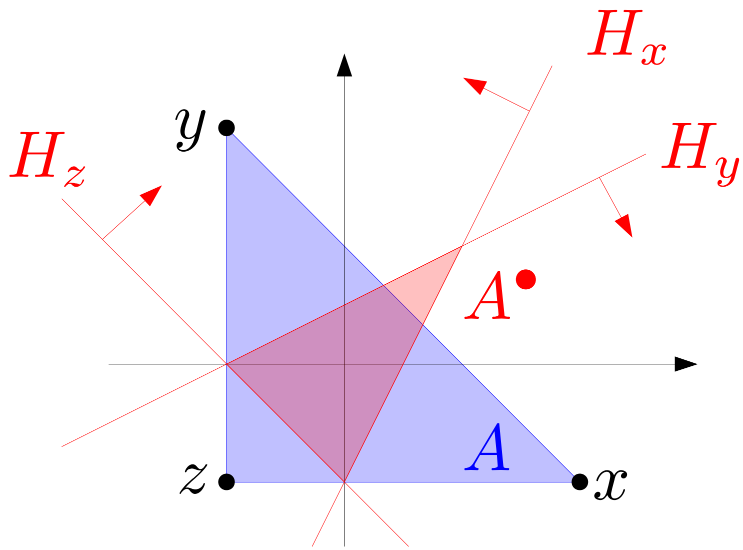
$\{x \in \mathbb{R}^2 \mid c^\mathsf{T} x = \gamma\}$

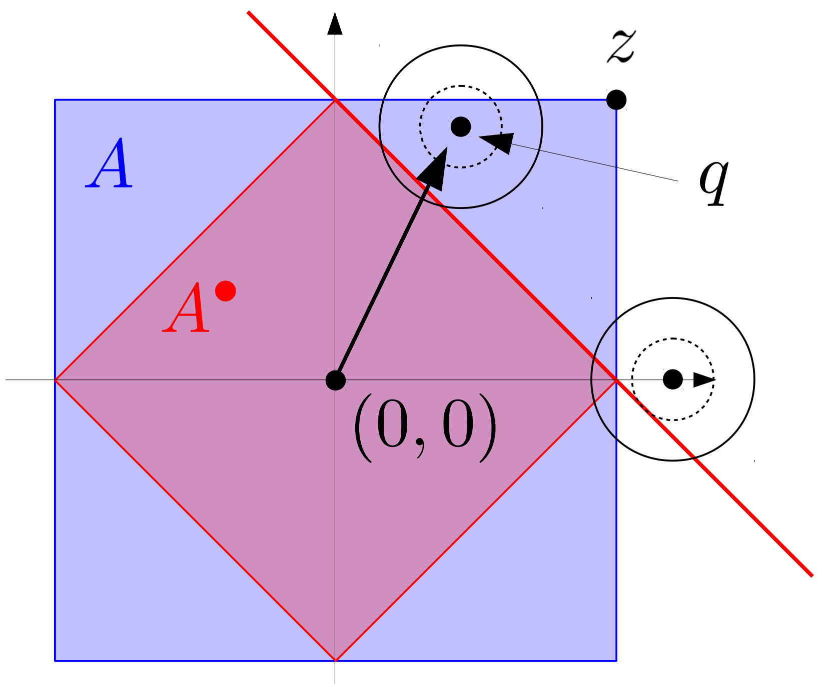


Proof sketch

3. Definition of "polar set"

$$A^\bullet := \{x \in \mathbb{R}^d \mid \forall\, a \in A.\ x^\mathsf{T} a \le 1\}$$





4. Reducing WOPT to ψ

$$H_z^1 := \{x \in \mathbb{R}^2 \mid x^\mathsf{T} z = 1\}$$

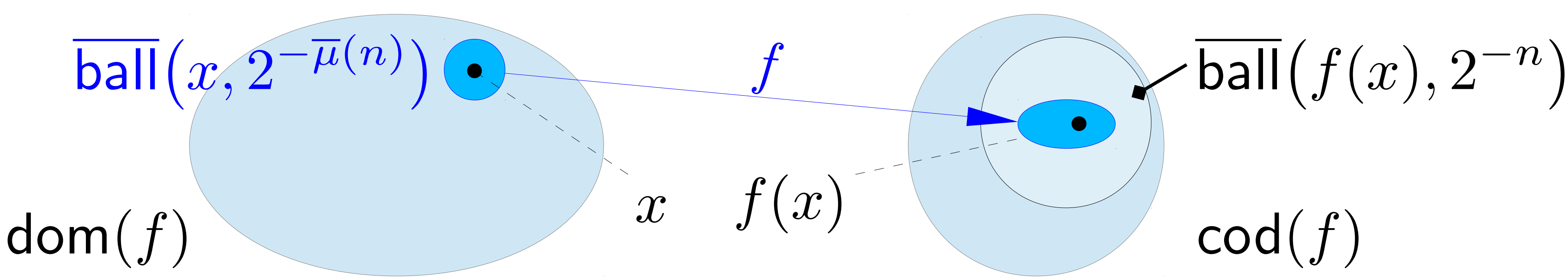
$\square$

Part II: Complexity of Function Inversion

Moduli: ... of **continuity** / ... of **unicity**

- Modulus of **continuity** $\overline{\mu}(n)$:

$$|x-y| \le 2^{-\overline{\mu}(n)} \implies |f(x)-f(y)| \le 2^{-n}$$



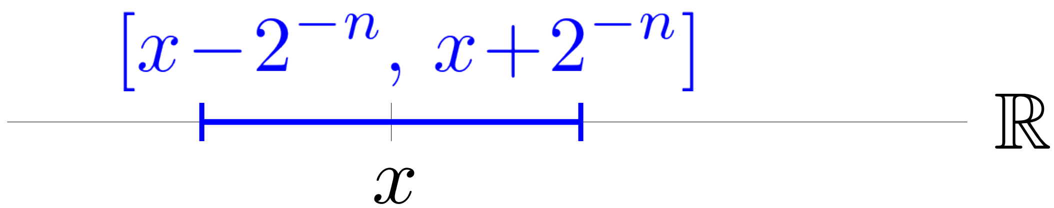
- Modulus of **unicity** $\underline{\mu}(n)$ (also: **inverse modulus**):

$$|x-y| > 2^{-n} \implies |f(x)-f(y)| \le 2^{-\underline{\mu}(n)}$$

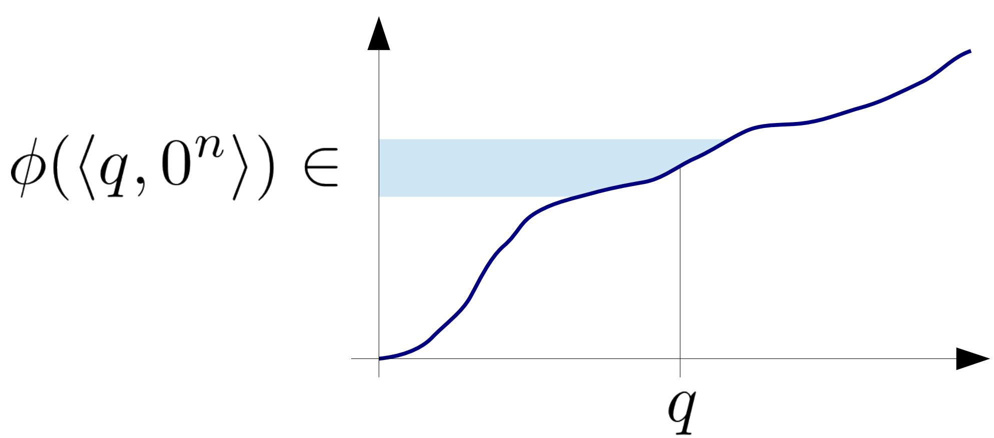
$$|f(x)-f(y)| \le 2^{-\underline{\mu}(n)} \implies |x-y| \le 2^{-n}$$

Representing Reals and Functions

- ϕ is $\boldsymbol{\rho}$-name of $x \in \mathbb{R}$, iff $\left|x - \phi(\langle q, 0^n \rangle)\right| \leq 2^{-n}$



- $\langle \phi, \overline{\mu} \rangle$ is $\boldsymbol{\delta}$-name of $f \in C[0{,}1]$, iff
  - $\left|f(q) - \phi(\langle q, 0^n \rangle)\right| \leq 2^{-n}$
  - $\overline{\mu}$ is modulus of continuity of f

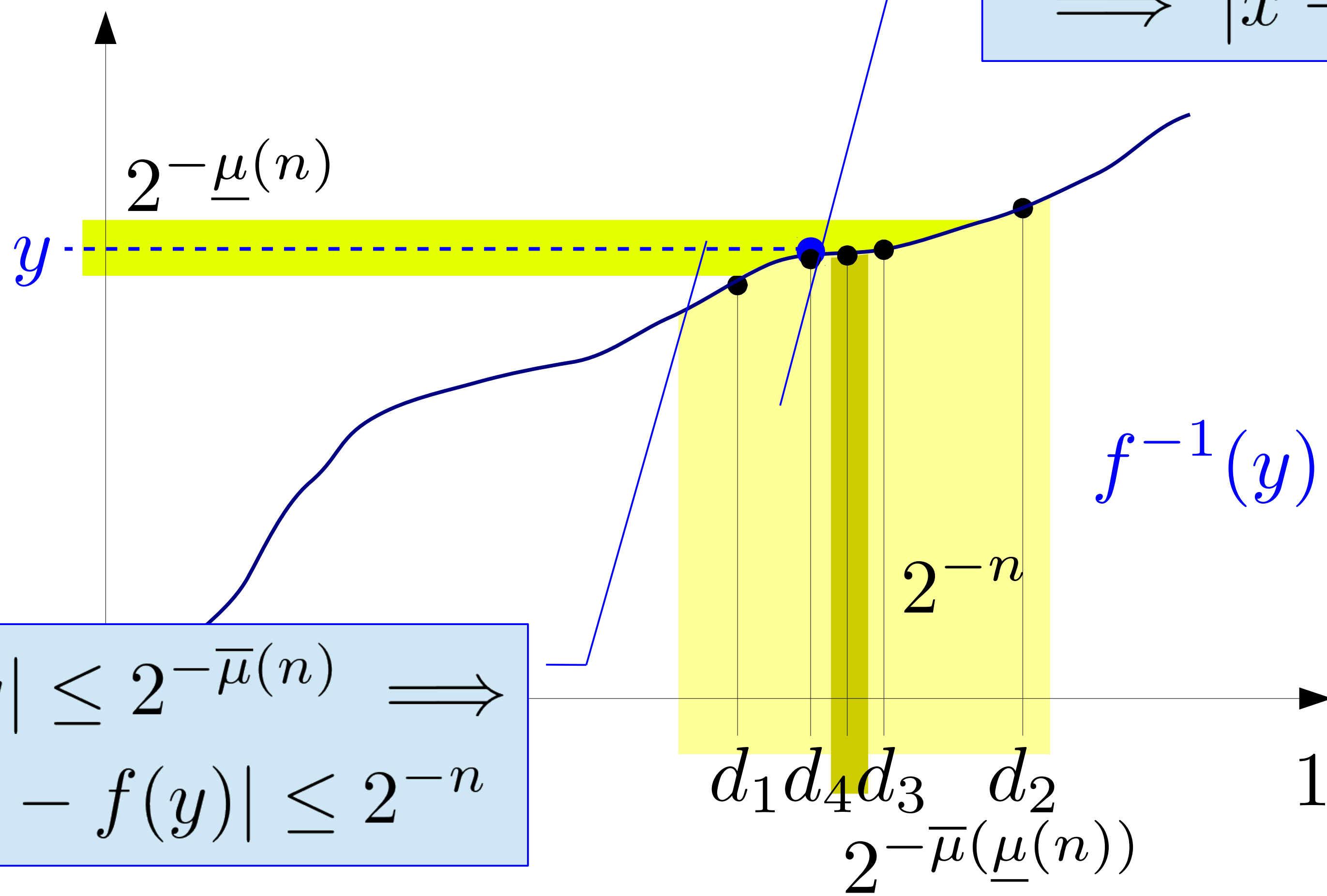


Inversion for functions $f\colon [0,1] \rightarrowtail \mathbb{R}$

Computation depends on both moduli:

Complexity: $\mathcal{O}\big(\overline{\mu}(\underline{\mu}(n))\big)$

$$|f(x) - f(y)| \leq 2^{-\underline{\mu}(n)} \implies |x - y| \leq 2^{-n}$$



$$f^{-1}(y) \in [d_* \pm 2^{-n}]$$

$$|x - y| \leq 2^{-\overline{\mu}(n)} \implies |f(x) - f(y)| \leq 2^{-n}$$

Inversion: $f \longmapsto f^{-1}$ on range(f)

Fact [Ko'91, Thm. 4.26]

P $\neq$ UP $\implies \exists\, f\colon [0,1]^2 \rightarrowtail [0,1]^2$, f poly-time, $\underline{\mu} \in \mathbb{N}[X]$

: $\boxed{f^{-1} \text{ **not** poly-time}}$

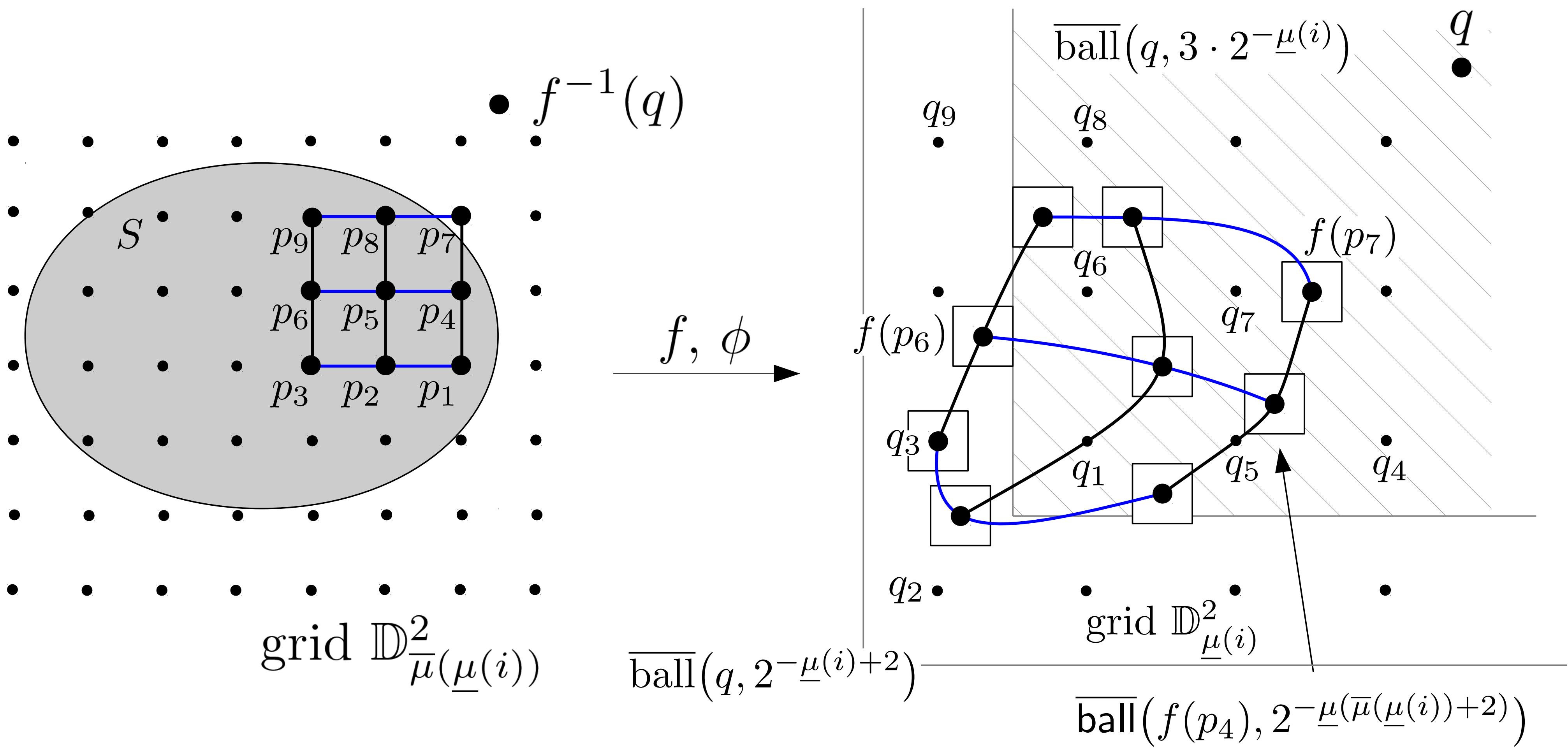
Characterization of UP [Ko'85, Grollman/Selman'88]:

$\phi\colon \Sigma^* \to \Sigma^*$: "easy" to compute, but "hard" to invert

P $\subseteq \boxed{\text{UP}} \subseteq$ NP, but neither P $\stackrel{?}{=}$ UP

nor NP $\stackrel{?}{=}$ UP are known

Inversion: Proof idea for "low" complexity



$$\text{dom}(f) = [0,1]^2 \implies \mathcal{O}\left(n \cdot 2^{\underline{\mu}(\overline{\mu}(\underline{\mu}(n))) - \underline{\mu}(n)}\right) \text{ points to check}$$

$\rightarrow$ for Lipschitz functions: $\mu(n) = n + \log L$

ご清聴ありがとうございました

... and have a nice prevening. :)

... translation by courtesy of japanese.stackexchange.com