Recursion schemes for P, NP and Pspace

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P, NP, Pspace: models of computation

- Model of computation
  - Ptime: Deterministic TM;
  - NP: Non-deterministic TM;
  - Pspace: Alternating TM.
- Resource constraint: polynomial time.

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DTM

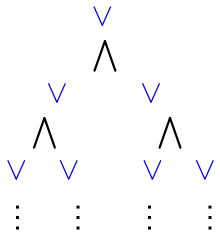
NTM and ATM

$$\begin{array}{c} c_0 \\ | \\ c_1 \\ | \\ c_2 \\ \vdots \end{array}$$

$$\begin{array}{cccccccc} & & & c_\epsilon & & & \\ & & & \bigwedge & & & \\ & c_0 & & & & c_1 & \\ & \bigwedge & & & & \bigwedge & \\ c_{00} & & c_{01} & & c_{10} & & c_{11} \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

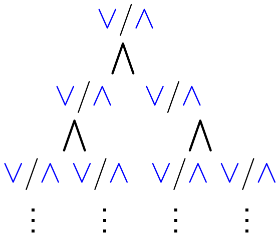
NP, Pspace: models of computation

NTM



final states 0 or 1

ATM



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Go back

Recursion-theoretic approach

- One works over $\mathbb{W} =< \epsilon, S_0, S_1 >$.
- Class of initial functions $\mathcal{I}$:
  - ϵ, S_0 and S_1 (the constructores of the algebra);
  - P (predecessor);
  - C (case distinction);
  - π_i^n (projections).

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$$[\mathcal{I}; \textbf{Composition}, \textbf{Recursion schemes}]$$

Recursion-theoretic approach

$$\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{Pspace}$$

$$\mathbf{FPtime} \subseteq \qquad\qquad\qquad\qquad \subseteq \mathbf{FPspace}$$

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- Bounded approach: machine independent, but resource dependent.
- Implicit approach: machine and resource independent.

Implicit recursion-theoretic approach

Functions have two sorts of input positions, normal and safe: $f(\bar{x}; \bar{y})$.

Input-sorted initial functions $\mathcal{SI}$:

- ϵ, $S_0(;x)$ and $S_1(;x)$ (the constructores of the algebra);
- $P(;x)$ (predecessor);
- $C(;x,y,z_0,z_1)$ (case distinction);
- $\pi_i^{m;n}$ (projections over both input sorts).

Input-sorted composition SC: $f(\bar{x}; \bar{y}) = g(\bar{r}(\bar{x};) \,;\, \bar{s}(\bar{x}; \bar{y}))$.

$[\mathcal{SI}; \mathbf{SC},$ **Input-sorted Recursion** $]$

FPtime FNPtime FPspace

Implicit recursion-theoretic approach: **FPtime**

$\textbf{FPtime} = [\,\mathcal{SI}; \textbf{SC}, \textbf{SR}\,]$ (Bellantoni-Cook 1992)

SR **(Input-sorted recursion over $\mathbb{W}$)**: $\boxed{<<}$

$$
\begin{aligned}
f(\epsilon, \bar{x}; \bar{y}) &= g(\epsilon, \bar{x}; \bar{y}) \\
f(z0, \bar{x}; \bar{y}) &= h(z0, \bar{x}; \bar{y}, f(z, \bar{x}; \bar{y})) \\
f(z1, \bar{x}; \bar{y}) &= h(z1, \bar{x}; \bar{y}, f(z, \bar{x}; \bar{y}))
\end{aligned}
$$

Example

Implicit recursion-theoretic approach:

$\mathbf{FPtime} = [\,\mathcal{SI}; \mathbf{SC}, \mathbf{SR}\,]$ (Bellantoni-Cook 1992)
$\mathbf{FPspace} = [\,\mathcal{SI}; \mathbf{SC}, \mathbf{STR}\,]$ (Mamino and O.)

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STR: Example

$$
\begin{aligned}
f(\epsilon, \bar{x}; \bar{y}, p) &= g(\epsilon, \bar{x}; \bar{y}, p) \\
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f(z1, \bar{x}; \bar{y}, p) &= h(z1, \bar{x}; \bar{y}, f(z, \bar{x}; \bar{y}, p0), f(z, \bar{x}; \bar{y}, p1))
\end{aligned}
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Implicit recursion-theoretic approach: **FNPtime**

$$\mathbf{FPtime} \subseteq \underbrace{\mathbf{FPtime} \cup \mathbf{NP}}_{\mathit{FNPtime}} \subseteq \mathbf{FPspace}$$

$$\mathbf{FNPtime} = [\,\mathbf{FPtime}_{BC}; ?, \quad]$$

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COMP_{BC}: $f(\bar{x}) = h(\bar{g}(\bar{x}))$ with $\bar{g} \in \mathrm{FPtime}_{BC}$

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$$\begin{array}{ccc} | & & | \\ \mathbf{SR} & \mathbf{?} & \mathbf{STR} \end{array}$$

$$\mathbf{FNPtime} = [\,\mathbf{FPtime}_{BC}; \mathbf{COMP}_{BC}, ?\,]$$

STR:

$$\begin{aligned}
f(\epsilon, \bar{x}; \bar{y}, p) &= g(\epsilon, \bar{x}; \bar{y}, p) \\
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SR	**?**	**STR**

$$\mathbf{FNPtime} = [\,\mathbf{FPtime}_{BC}; \mathbf{COMP}_{BC}, \mathbf{STR}[\vee]\,] \qquad \text{(O 2011)}$$

STR[∨]:

$$\begin{aligned}
f(\epsilon, \bar{x}; \bar{y}, p) &= g(\epsilon, \bar{x}; \bar{y}, p)\\
f(z0, \bar{x}; \bar{y}, p) &= \vee(\qquad ; \quad f(z, \bar{x}; \bar{y}, p0), f(z, \bar{x}; \bar{y}, p1))\\
f(z1, \bar{x}; \bar{y}, p) &= \vee(\qquad ; \quad f(z, \bar{x}; \bar{y}, p0), f(z, \bar{x}; \bar{y}, p1))
\end{aligned}$$

Implicit recursion-theoretic approach: **FNPtime**

STR[$\vee$]:

$$
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\end{aligned}
$$

Example:

$f(11; \epsilon)$ leads to $\vee(\vee(g(00), g(01)), \vee(g(10), g(11)))$

$$
\begin{array}{c}
\vee \\
\bigwedge \\
\vee \qquad \vee \\
\bigwedge \qquad\quad \bigwedge \\
g00 \;\; g01 \quad g10 \;\; g11
\end{array}
$$

Implicit recursion-theoretic approach: **FNPtime**

STR[$\vee$]:

$$f(\epsilon, \bar{x}; \bar{y}, p) = g(\epsilon, \bar{x}; \bar{y}, p)$$
$$f(z0, \bar{x}; \bar{y}, p) = \vee(\quad ; \quad f(z, \bar{x}; \bar{y}, p0), f(z, \bar{x}; \bar{y}, p1))$$
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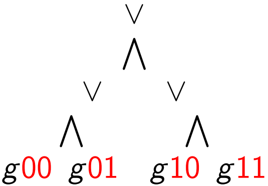

Only the addresses of the leaves are available.
All internal nodes have the same (disjunctive) label.

Implicit recursion-theoretic approach

Input-sorted composition SC: $f(\bar{x};\bar{y}) = g(\bar{r}(\bar{x};)\,;\,\bar{s}(\bar{x};\bar{y}))$.

If $F(x;y)$ is in the class then

- $f(x,y;) = F(x;y)$ is in the class;
- $f(;x,y) = F(x;y)$ is NOT in the class.

<<

Implicit recursion-theoretic approach: **FPtime**

SR:

$$f(\epsilon, \bar{x}; \bar{y}) = g(\epsilon, \bar{x}; \bar{y})$$
$$f(z0, \bar{x}; \bar{y}) = h(z0, \bar{x}; \bar{y}, f(z, \bar{x}; \bar{y}))$$
$$f(z1, \bar{x}; \bar{y}) = h(z1, \bar{x}; \bar{y}, f(z, \bar{x}; \bar{y}))$$

- Concatenation:
$$\oplus(\epsilon; x) = x$$
$$\oplus(y0; x) = S_0(; \oplus(y; x))$$
$$\oplus(y1; x) = S_1(; \oplus(y; x))$$

$\oplus(y, x;)$ is in the class.
$\oplus(; y, x)$ is NOT in the class.

Implicit recursion-theoretic approach: **FPtime**

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- exp such that $|\exp(z)| = 2^{|z|}$

$$\exp(\epsilon) = 1$$
$$\exp(zi) = \oplus(\exp(z), \exp(z))$$

Implicit recursion-theoretic approach: **FPtime**

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PROBLEM!

Implicit recursion-theoretic approach: **FPtime**

SR (Input-sorted recursion over $\mathbb{W}$**):**

$$\begin{aligned} f(\epsilon, \bar{x}; \bar{y}) &= g(\epsilon, \bar{x}; \bar{y}) \\ f(z0, \bar{x}; \bar{y}) &= h(z0, \bar{x}; \bar{y}, f(z, \bar{x}; \bar{y})) \\ f(z1, \bar{x}; \bar{y}) &= h(z1, \bar{x}; \bar{y}, f(z, \bar{x}; \bar{y})) \end{aligned}$$

Example: $f(11;)$ leads to $h(11; h(1; g(\epsilon;)))$

$$\begin{array}{c} h \\ | \\ h \\ | \\ g \end{array}$$

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$$
\begin{array}{c}
h \\
| \\
h \\
| \\
g
\end{array}
$$

SR reproduces the sequential structure of deterministic computations.

Implicit recursion-theoretic approach: **FPspace**

STR:

$$f(\epsilon, \bar{x}; \bar{y}, p) = g(\epsilon, \bar{x}; \bar{y}, p)$$
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Example: $f(11; \epsilon)$ leads to
$h(11; h(1; g(\epsilon; 00), g(\epsilon; 01)), h(1; g(\epsilon; 10), g(\epsilon; 11)))$.

$$\begin{array}{ccccccc}
 & & & h & & & \\
 & & & \bigwedge & & & \\
 & & h & & h & & \\
 & \bigwedge & & & & \bigwedge & \\
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\end{array}$$

Implicit recursion-theoretic approach: **FPspace**

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The mentioned input is the **pointer**, and it gives the **address from the root of the tree to the leaves**.

<<

STR trivially **extends SR**.