Expressive Power and Algorithms

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Introduction

Motivations

- Are “complicated” programming constructions really useful? We’re still Turing-complete without them.
  High-order? Non-determinism? Co-arity? Multiple tapes? Large alphabets? . . .

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- Are "complicated" programming constructions really useful? We're still Turing-complete without them.
  High-order? Non-determinism? Co-arity? Multiple tapes? Large alphabets? . . .
- Which is "the best" ICC system for PTIME?
  MPO+QI? DLAL? *mwp*-polynomials? NSI? . . .
- How to even compare them?

Magic Trick!

Explaining the Trick

- bead = program
- color = computed function
- small bead = a given syntactical criterion

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- bead = program
- color = computed function
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Each set of "programs" computes the same set of "functions". But the first one has more things (small black beads) and only the filter can reveal it.

There is no color- and size-preserving function from the first set to the second.

Life without Cons

First order functional programs are Turing-complete. Why using high-order?

- Libraries

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- Second order CONS free: EXPTIME
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High order is "more expressive" than first order, but Turing-completeness hide this.

Expressive Power

Compilation

- Programming language: $\mathcal{P}$ $\supset L, M$ (syntactic)

L

M

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$$\begin{tikzcd}
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Hard to prove

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- Computable functions: $\mathcal{C}$
- Semantics: $\mathcal{P} \xrightarrow{[\![\bullet]\!]} \mathcal{C}$ $[\![L]\!] = A = B = [\![M]\!]$
- Compilation $\varphi : L \to M$, $[\![\varphi(\mathtt{p})]\!] = [\![\mathtt{p}]\!]$ Semantics preserving

$$
\begin{tikzcd}
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Hypothesis

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Conclusion

There is no filter-preserving compilation: $\mathtt{p} \in F \Rightarrow \varphi(\mathtt{p}) \in F$

Conversely, if there exists a filter preserving compilation, then $A' = B'$.

Expressiveness

$$\begin{tikzcd}
& L' \arrow[rr, two heads, "\llbracket\bullet\rrbracket"] \arrow[dl, hook', "\iota"] \arrow[dd, dotted, "\varphi"] & & A' \arrow[dl, hook', "\iota"] \arrow[dd, "id"'] \\
L \arrow[rr, two heads, "\llbracket\bullet\rrbracket"] \arrow[dd, "\varphi"'] & & A \arrow[dd, "id"] & \\
& M' \arrow[rr, dotted, two heads, "\llbracket\bullet\rrbracket"] \arrow[dl, dotted, hook', "\iota"] & & B' \arrow[dl, hook', "\iota"] \\
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$\mathtt{p} \in L' = L \cap F$
$\Rightarrow \varphi(\mathtt{p}) \in M \cap F = M'$

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$$\begin{tikzcd} & \mathrm{M'} \arrow[rr, two heads, dotted, "\llbracket\bullet\rrbracket"] \arrow[dl, hook, dotted, "\iota"] & & \mathrm{B'} \arrow[dl, hook, "\iota"] \\ \mathrm{M} \arrow[rr, two heads, "\llbracket\bullet\rrbracket"] & & \mathrm{B} & \end{tikzcd}$$

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$\mathtt{p} \in L' = L \cap F$

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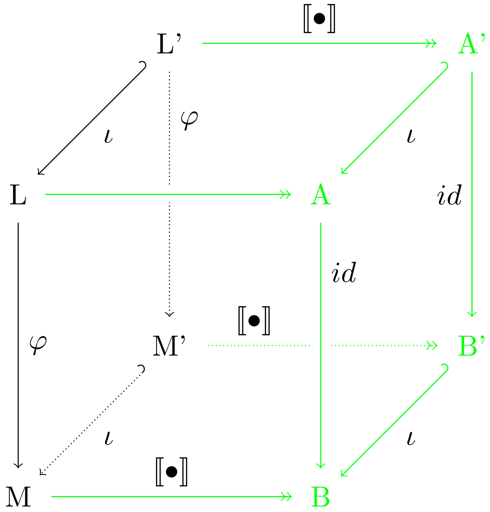
Canonical injection

Expressiveness



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Compilation

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$\Rightarrow \varphi(\mathtt{p}) \in M \cap F = M'$

Canonical injection

$f \in A'$

$\mathtt{p} \in L', [\![\mathtt{p}]\!] = f$

$\mathtt{q} = \varphi(\mathtt{p}) \in M', [\![\mathtt{q}]\!] = f$

$f \in B'$

High Order

$$
\begin{tikzcd}
& (2^{\mathrm{nd}})\ \backslash\mathrm{CONS} \arrow[ld, hook', "\iota"] \arrow[dd, dotted, "\varphi"] \arrow[rr, two heads, "\llbracket\bullet\rrbracket"] & & \textsc{EXPTime} \arrow[ld, hook', "\iota"'] \arrow[dd, "id"] \\
(2^{\mathrm{nd}}) \arrow[rr, two heads, "\llbracket\bullet\rrbracket"] \arrow[dd, "\varphi"'] & & \mathcal{C} \arrow[dd, "id"] & \\
& (1^{\mathrm{st}})\ \backslash\mathrm{CONS} \arrow[ld, dotted, hook', "\iota"] \arrow[rr, dotted, two heads, "\llbracket\bullet\rrbracket"] & & \textsc{PTime} \arrow[ld, hook', "\iota"] \\
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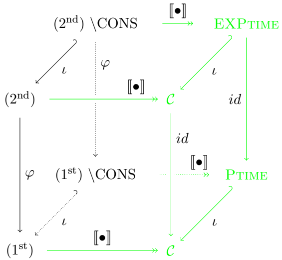
$\mathcal{P}$ = high order TRS
$L = (2^{\mathrm{nd}})$ $M = (1^{\mathrm{st}})$ $F = \backslash\mathrm{CONS}$

High Order



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$[\![(2^{\text{nd}})]\!] = \mathcal{C} = [\![(1^{\text{st}})]\!]$

$[\![(2^{\text{nd}})\backslash\text{CONS}]\!] = \text{EXPTIME}$

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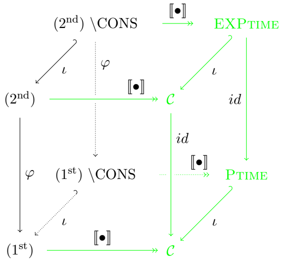
High Order



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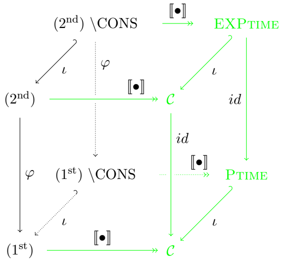
EXPTIME $\neq$ PTIME, hence there exists no compilation from (2^{nd}) to (1^{st}) preserving $\setminus$CONS.

High Order



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High order programs are more expressive than first order ones.

Non-Determinism

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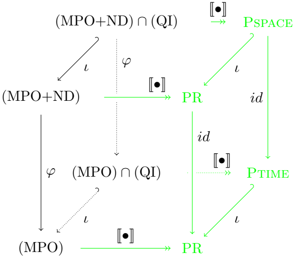

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Non-deterministic programs are more expressive than first order ones.

Algorithms

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Algorithms lay somewhere between programs and functions.

$$\begin{tikzcd} \mathcal{P} \arrow[r, two heads, "|\bullet|"] \arrow[rr, bend right, two heads, "[\![\bullet]\!]"'] & \mathcal{Z} \arrow[r, two heads, "[\bullet]"] & \mathcal{C} \end{tikzcd}$$

$$[|\mathtt{p}|] = [\![\mathtt{p}]\!]$$

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Transforming algorithms

Hypothesis

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Algorithm transformation: $\tau : X \to Y$

Conclusion

There is no filter-preserving transformation of algorithms:
$x \in |F| \Rightarrow \tau(x) \in |F|$

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Conversely, if there exists a filter preserving transformation (incl. identity), then $A' = B'$.

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[Filtering]

- There are things written in beads of the first set that are not written in beads of the second (those in small black beads).

The big picture

$$
\begin{tikzcd}
 & L' \arrow[rr, two heads, "\lvert \bullet \rvert"] \arrow[dl, hook, "\iota"] & & X' \arrow[rr, two heads, "{[\bullet]}"] \arrow[dl, hook, "\iota"] \arrow[dd, dotted, "\tau"] & & A' \arrow[dl, hook, "\iota"] \arrow[dd, "id"] \\
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High order

$$
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\end{tikzcd}
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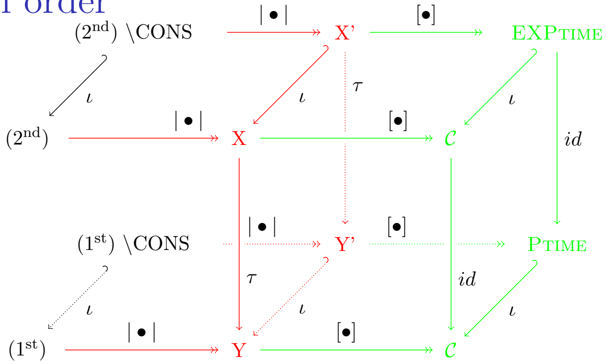
Algorithms respect cons-free.

High order



Algorithms respect cons-free.

There is no transformation from high order algorithms to first order algorithm *preserving cons-free.*

There are cons-free high-order algorithms with no first order counterpart.

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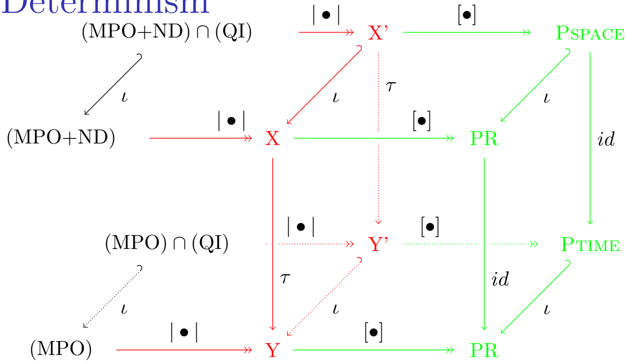
Non Determinism

$$\begin{tikzcd}
 & (\mathrm{MPO+ND}) \cap (\mathrm{QI}) \arrow[rr, two heads, "|\bullet|"] \arrow[dl, hook', "\iota"] & & \mathrm{X'} \arrow[rr, two heads, "{[\bullet]}"] \arrow[dl, hook', "\iota"] \arrow[dd, dotted, "\tau"] & & \textsc{Pspace} \arrow[dl, hook', "\iota"] \arrow[dd, "id"] \\
(\mathrm{MPO+ND}) \arrow[rr, two heads, "|\bullet|"] & & \mathrm{X} \arrow[rr, two heads, "{[\bullet]}"] \arrow[dd, "\tau"] & & \mathrm{PR} \arrow[dd, "id"] & \\
 & (\mathrm{MPO}) \cap (\mathrm{QI}) \arrow[rr, dotted, two heads, "|\bullet|"] \arrow[dl, dotted, hook', "\iota"] & & \mathrm{Y'} \arrow[rr, dotted, two heads, "{[\bullet]}"] \arrow[dl, dotted, hook', "\iota"] & & \textsc{Ptime} \arrow[dl, hook', "\iota"] \\
(\mathrm{MPO}) \arrow[rr, two heads, "|\bullet|"] & & \mathrm{Y} \arrow[rr, two heads, "{[\bullet]}"] & & \mathrm{PR} &
\end{tikzcd}$$

Algorithms respect QI.

Non Determinism



Algorithms respect QI.

There is no transformation from non-deterministic algorithms to first order algorithm *preserving Quasi-Interpretations*.
There are Non-deterministic algorithms admitting a Quasi-Interpretation with no deterministic counterpart.

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Future Works

Other Examples

- The framework is generic and could be applied to many cases.
- But we still need to find a good filter and prove:
  $[\![L]\!] = [\![M]\!]$
  $[\![L']\!] \neq [\![M']\!]$

Other Examples

- The framework is generic and could be applied to many cases.
- But we still need to find a good filter and prove:
  $[\![L]\!] = [\![M]\!]$
  $[\![L']\!] \neq [\![M']\!]$
- Fortunately, we have 20 years of ICC with lot of results of this kind to use!
- Linear Logic? Imperative programs? ...

Cross-language comparison

Can we compare systems for imperative programs with Linear Logic systems?

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- The key to the proof is that the filter is “the same” in both cases.
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  Especially, $[\![F]\!] = [\![G]\!]$ is not sufficient.

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- When dealing with algorithms, if we have two filters F and G, we need $|F| = |G|$
  Especially, $[\![F]\!] = [\![G]\!]$ is not sufficient.
- But without knowing what an algorithm is, we cannot prove that.

We have to find another way to circumvent this.

Comparing Systems for PTIME

	(1^{st}) \CONS	Interp.	MPO+QI

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- Experiments suggest that it's easier to write MPO+QI programs than (1^{st}) \CONS ones.
- Adding Non-determinism reveals a jump in expressiveness.
- There was more "things" (algorithms) in MPO+QI but we need the Non-determinism to make them appear.

Chemistry

A chemical version of this:

Conclusion

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- Generic framework.
- Can tell something about algorithms, under "reasonable" assumptions (or at least we can argue about the assumptions).
- Reuse 20 years of ICC to get more results.
- Can still be extended for more interesting results?
- A step toward a theory of algorithms?