Amortised Resource Analysis and Typed Term Rewriting (work in progress)

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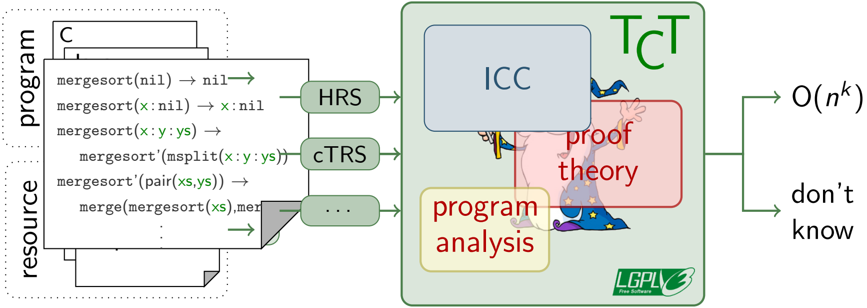
Overview

- Motivation
- Amortised Resource Analysis
- Amortised Complexity Analysis for Rewrite Systems
- Type System for Rewrite Systems
- Work in Progress

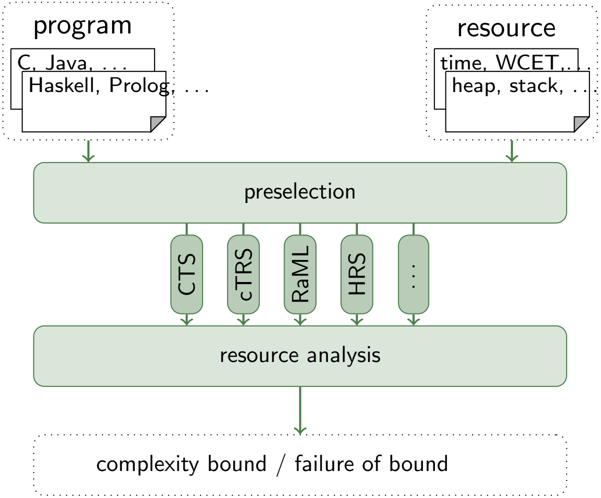
Tyrolean Complexity Tool

(runtime) complexity analyser for term rewrite systems (TRSs)

http://cl-informatik.uibk.ac.at/software/tct



Vision



implies a quest for the relative expressibility of techniques

Example (Division in RaML)

```
minus : (nat,nat) -> nat
minus (m,n) = match n with
      | 0 -> m
      | S n' -> match m with
                | 0 -> 0
                | S m' -> minus (m',n');

quot : (nat,nat) -> nat
quot (m,n) = match m with
      | 0 -> 0
      | S m' -> match n with
                | 0 -> 0
                | S n' -> (quot ( minus(m',n'),n )) + 1;
```

J. Hoffmann, K. Aehlig, and M. Hofmann.
Multivariate amortized resource analysis.
ACM Trans. Program. Lang. Syst., 34(3):14, 2012.

Comparison: The Easy Way

input	RaML	T_CT	actual behaviour
isortlist	$8n^2m + 8n^2 - 8nm + 4n + 3$	$O(k^2)$	$O(n^2m)$
nub	$6n^2m + 9n^2 - 6nm + 3n + 3$	$O(k^2)$	$O(n^2m)$
matrixMultList	$28xyzn + 18xyz + 12xy + \dots$	$O(k^3)$	$O(xyzn)$
dyad	$10nx + 14n + 3$	$O(k^2)$	$O(nx)$
lcs	$39nx + 6x + 21n + 19$	$O(k^2)$	$O(nx)$
subtrees	$8n^2 + 19n + 3$	$O(k^2)$	$O(n^2)$
eratos	$8n^2 + 4n + 3$	$O(k^2)$	$O(n^2)$
splitandsort	$21n^2 + 37n + 9$	$O(k^2)$	$O(n^2)$
triples	$6n^3 - 10n^2 + 24n + 3$	TO	$O(n^3)$

Observations

- the library tuples is the only library T_CT cannot handle
- typically execution times of T_CT are horrible

A Closer Look

Example

```
minus : (nat,nat) -> nat
[...]

quot : (nat,nat) -> nat
[...]
```

```
$ raml --lp_solve analyse eval-steps 1 div.raml

The number of evaluation steps consumed by quot is at most:
          18.0*n + 3.0
where
   n is the value of the first component of the input
   m is the value of the second component of the input
```

Example

$$\mathsf{minus}(x,0) \to x \qquad \mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}(x,y)$$
$$\mathsf{quot}(0,\mathsf{s}(y)) \to 0 \qquad \mathsf{quot}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{s}(\mathsf{quot}(\mathsf{minus}(x,y),\mathsf{s}(y)))$$

Observation

consider the derivation from $\mathbf{4} \div \mathbf{2}$:

$$\begin{aligned}\mathsf{quot}(\mathbf{4},\mathbf{2}) &\to \mathsf{s}(\mathsf{quot}(\mathsf{minus}(\mathbf{3},\mathbf{1}),\mathbf{2})) \\ &\to \mathsf{s}(\mathsf{quot}(\mathsf{minus}(\mathbf{2},0),\mathbf{2}) \to \mathsf{s}(\mathsf{quot}(\mathbf{2},\mathbf{2})) \to \cdots\end{aligned}$$

Conclusion

- 2nd argument of quot never changes
- interpretation $\mathcal{A}$ monotonic in the first argument suffices

Example (cont'd)

$0_{\mathcal{A}} = 0 \quad \mathsf{s}_{\mathcal{A}}(x) = x + 3 \quad \mathsf{minus}_{\mathcal{A}}(x,y) = x + 1 \quad \mathsf{quot}_{\mathcal{A}}(x,y) = 2x + 1$

Univariate Polynomial Potential

Definition

annotated types are given by the following grammar:

$$A ::= \mathsf{Int} \mid A \times \cdots \times A \mid L^{\vec{p}}(A) \qquad F ::= A_1 \times \cdots \times A_n \xrightarrow[q]{p} A$$

- $\vec{p} = (p_1, \ldots, p_k)$ with $p_i \in \mathbb{Q}^+, k \geqslant 1$
- $\vec{p}$ is called resource annotation (for list types)

Example

let $v = [v_1, \ldots, v_n]$ denote an integer list, then

$$\Phi(v\colon L^{\vec{p}}(\mathsf{Int})) = \sum_{i=1}^{n} p_i \cdot \binom{n}{i}$$

Constructor Term Rewrite Systems

Definition

a TRS is a finite set of rules of the following form:

$$f(c_1(\overline{x_1}), \ldots, c_n(\overline{x_n})) \to r$$

- $\forall\ i$: c_i is a constructor
- fixed set of constructors $= \{\mathsf{nil}\} \cup \mathbb{Z} \cup \{\mathsf{pair}, ::, \mathsf{op}\}$
- we study completely defined, orthogonal, constructor TRSs, with nesting of constructors is prohibited on a fixed set of constructors
- I think everything but constructor can be dropped

Definition

$$\mathsf{rc}(n) := \max\{\mathsf{dh}(t) \mid t \text{ is argument normalised and } |t| \leqslant n\}$$

Definition (operational semantics (subset))

$$\frac{x\sigma = v}{\sigma \left|\frac{m}{m}\right. x \Rightarrow v} \qquad \frac{y\sigma = \mathit{head} \quad ys\sigma = \mathit{tail}}{\sigma \left|\frac{m}{m}\right. y :: ys \Rightarrow \mathit{head} :: \mathit{tail}}$$

$$\frac{f(l_1, \dots, l_n) \to r \in \mathcal{R} \quad \exists \tau\ \forall i\colon x_i\sigma = l_i\tau \quad \sigma \uplus \tau \left|\frac{m}{m'}\right. r \Rightarrow v}{\sigma \left|\frac{m+1}{m'}\right. f(x_1, \dots, x_n) \Rightarrow v}$$

$$\frac{\sigma \left|\frac{m}{m_1}\right. t_1 \Rightarrow v_1 \quad \cdots \quad \sigma \left|\frac{m_{n-1}}{m_n}\right. t_n \Rightarrow v_n \quad \sigma \uplus \rho \left|\frac{m_n}{m'}\right. f(x_1, \dots, x_n) \Rightarrow v}{\sigma \left|\frac{m}{m'}\right. f(t_1, \dots, t_n) \Rightarrow v}$$

- $\rho := \{x_1 \mapsto v_1, \dots, x_n \mapsto v_n\}$
- σ, τ, and ρ are normalised

Lemma

$\sigma \left|\frac{m}{m'}\right. f(x_1, \dots, x_n) \Rightarrow t$ *iff* $\mathrm{dh}(f(x_1\sigma, \dots, x_n\sigma), \to_{\mathcal{R}}) = m - m'$

Definition

signature $\mathcal{F}$ maps constructors and (defined) function symbols to annotated types

1. constructors are mapped to data types
2. function symbols are mapped to first-order types

Definition

the additive shift of $\vec{p} = (p_1, \ldots, p_k)$ is defined as:

$$\lhd(\vec{p}) = (p_1 + p_2, p_2 + p_3, \ldots, p_{k-1} + p_k, p_k)$$

Definition

- duplication of variables is governed by the following operator

$$\curlyvee(A \,|\, A_1, A_2)$$

- $\curlyvee(A \,|\, A_1, A_2)$ states resources $A_1 + A_2 =$ resources A

Typing Rules

Definition (context)

$$\frac{\Gamma_1 \mathrel{\big|\!\frac{p}{p_1}} t_1\colon A_1 \quad \cdots \quad \Gamma_n \mathrel{\big|\!\frac{p_{n-1}}{p_n}} t_n\colon A_n \quad \overset{\text{all } x_i \text{ are fresh}}{x_1\colon A_1, \dots, x_n\colon A_n \mathrel{\big|\!\frac{p_n}{q}} f(x_1, \dots, x_n)\colon C}}{\Gamma_1, \dots, \Gamma_n \mathrel{\big|\!\frac{p}{q}} f(t_1, \dots, t_n)\colon C}$$

Definition (application)

$$\frac{\mathcal{F}(f) = A_1 \times \cdots \times A_n \xrightarrow[q]{p} C}{x_1\colon A_1, \dots, x_n\colon A_n \mathrel{\big|\!\frac{p}{q}} f(x_1, \dots, x_n)\colon C}$$

Definition (lists)

$$\frac{A \text{ a data type} \quad \vec{p} = (p_1, \dots, p_k)}{x\colon A, xs\colon L^{\lhd(\vec{p})} \mathrel{\big|\!\frac{p_1}{0}} x :: xs\colon L^{\vec{p}}(A)}$$

Definition (variables & structural)

$$x\colon A \overset{0}{\underset{0}{\vdash}} x\colon A \qquad \frac{\Gamma \overset{p}{\underset{q}{\vdash}} t\colon C \quad p' \geqslant p \quad p' - p \geqslant q' - q}{\Gamma \overset{p'}{\underset{q'}{\vdash}} t\colon C}$$

$$\frac{\Gamma \overset{p}{\underset{q}{\vdash}} t\colon C}{\Gamma, x\colon A \overset{p}{\underset{q}{\vdash}} t\colon C} \qquad \frac{\Gamma, x\colon A_1, y\colon A_2 \overset{p}{\underset{q}{\vdash}} t[x,y]\colon C \quad \curlyvee(A \mid A_1, A_2)}{\Gamma, z\colon A \overset{p}{\underset{q}{\vdash}} t[z,z]\colon C}$$

Definition (simplified)

rule $f(c_1(x_1), \ldots, c_n(x_n)) \to r$ such that $\mathcal{F}(f) = A_1 \times \cdots \times A_n \xrightarrow[q]{p} C$ is well-typed, if

1. for all $i = 1, \ldots, n$: $x_i\colon B_i \overset{k_i}{\underset{0}{\vdash}} c_i(x_i)\colon A_i$, and
2. $x_1\colon B_1, \ldots, x_n\colon B_n \overset{p-1+\sum_{i=1}^{n} k_i}{\underset{q}{\vdash}} r\colon C$

Example

consider the TRS

$$\begin{aligned} \mathsf{append}(\mathsf{nil}, ys) &\to ys \\ \mathsf{append}(x :: xs, ys) &\to x :: \mathsf{append}(xs, ys) \end{aligned}$$

where $\mathcal{F}(\mathsf{append}) = L^{(1,1,0)}(\mathsf{Int}) \times L^{(0,0,0)}(\mathsf{Int}) \xrightarrow[0]{1} L^{(0,0,0)}(\mathsf{Int})$

we show well-typedness for each rule:

1. $\mathsf{append}(\mathsf{nil}, ys) \to ys$
2. $\mathsf{append}(x :: xs, ys) \to x :: \mathsf{append}(xs, ys)$

$$\dfrac{\dfrac{h\colon \mathsf{Int}, t\colon L^{(0,0,0)} \mid\!\frac{0}{0}\; h :: t\colon L^{(0,0,0)}}{x\colon \mathsf{Int} \mid\!\frac{1}{1}\; x\colon \mathsf{Int}} \qquad \dfrac{\mathcal{F}(\mathsf{append}) = L^{(1,1,0)} \times L^{(0,0,0)} \xrightarrow[0]{1} L^{(0,0,0)}}{xs\colon L^{(1,1,0)}, ys\colon L^{(0,0,0)} \mid\!\frac{1}{0}\; \mathsf{append}(xs, ys)\colon L^{(0,0,0)}}}{x\colon \mathsf{Int}, xs\colon L^{(1,1,0)}, ys\colon L^{(0,0,0)} \mid\!\frac{1-1+1}{0}\; x :: \mathsf{append}(xs, ys)\colon L^{(0,0,0)}}$$

Definition

let v be a value with type A, its potential is defined as follows

$$\Phi(v \colon A) := \begin{cases} 0 & \text{if } A = \mathsf{Int}\} \text{ of } v = \mathsf{nil} \\ \Phi(v_1 \colon B) + \Phi(v_2 \colon C) & \text{if } A = B \times C \\ p_1 + \Phi(v_1 \colon B) + \Phi(v_2 \colon L^{\lhd(\vec{p})}(B)) & \text{if } A = L^{\vec{p}}(B) \text{ and } v = (v_1 :: v_2) \end{cases}$$

Lemma

1. *let $\mathcal{R}$ and σ be well-typed*
2. *suppose* $\Gamma \left|\frac{p}{q}\right. f(x_1, \ldots, x_n) \colon A$
3. *suppose* $\sigma \left|\frac{m}{m'}\right. f(x_1, \ldots, x_n) \Rightarrow v$

then

$$\Phi(\sigma \colon \Gamma) + (p - q) - \Phi(v \colon A) \geqslant m - m'$$

hence $\Phi(\sigma \colon \Gamma) \geqslant \mathrm{dh}(f(x_1, \ldots, x_n)\sigma, \to_{\mathcal{R}})$

Interpretations for Typed TRSs

Definition (interpretation of function symbols)

- consider $A = B \times C$, define $\gamma(\mathsf{pair})(x, y) = x + y$
- consider $A = L^{\vec{p}}$, $\vec{p} = (p_1, \ldots, p_k)$, define

$$\gamma(::)(x, xs) = p_1 + x + xs$$

- consider function symbol f, $\mathcal{F}(f) = A_1 \times \cdots \times A_n \xrightarrow[q]{p} C$, define

$$\gamma(f)(x_1, \ldots, x_n) = x_1 + \cdots + x_n + (p - q)$$

- otherwise $\gamma(c)(\overline{x}) = 0$

Example

the datatype $L^{1,1,0}$ induces interpretation $\gamma(::)(x, xs) = x + xs + 1$

Definition

interpretations for types are extended to interpretations of terms:

$$[\![f(t_1,\dots,t_n)\colon A]\!] := \gamma(f)([\![t_1\colon A_1]\!],\dots,[\![t_n\colon A_n]\!])$$

Lemma

- *suppose $\mathcal{R}$ and σ are well-typed*
- *let γ be induced by the well-typing of $\mathcal{R}$*

then $\forall\ l \to r$: $[\![l\sigma\colon C]\!] > [\![r\sigma\colon C]\!]$

Proof Sketch.

for an extended definition of Φ, we have $[\![t\colon A]\!] = \Phi(t\colon A)$ ∎

Theorem

Let $\mathcal{R}$ be well-typed, $\exists$ typed polynomial interpretation (over $\mathbb{Q}^+$) orienting $\mathcal{R}$

Unrestricted Signature

1. would be more natural for TRSs
2. "*rek Typen sind eigentlich kein pb, gehen so wie baeume*" (MH)

Linear Potential (Jost et al., 2009)

1. type system can be rephrased for the linear potential method; restriction to specific signature can be dropped
2. let $\mathcal{R}$ be well-typed according to the linear type system, then there exists a linear interpretation over $\mathbb{Q}^+$ that orients $\mathcal{R}$

Multivariate Analysis (Hoffmann et al., 2011

1. it seems possible to adapt the presented type system to the multivariate analysis of RaML programs to rewrite systems
2. let $\mathcal{R}$ be well-typed according to this type system then there may exists a polynomial interpretation over $\mathbb{Q}^+$ that orients $\mathcal{R}$

Thank You for Your Attention!