Higher order complexity and application in computable analysis

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Background on higher order complexity

Higher order strategies

Higher order Turing machines

Computable analysis

Perspectives

Higher order complexity today:

- ✓ Order 1 ($\mathbb{N} \to \mathbb{N}$) computability and complexity
- ✓ Order 2 ($(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$): Oracle Turing Machines (OTM)
- ✗ Order 3 and above:
  - BFF
  - but it is a far smaller class than the continuous functionals (contrary to orders 1 and 2).
  - no satisfying machine model
  - no general notion of complexity
  - several (incomparable) notions of computability, but one natural and robust class: the Kleene-Kreisel functionals.
- How to define a general and meaningful notion of complexity at all finite types?

Basic Feasible Functionals (BFF)

Definition

PV^ω = simply-typed λ-calculus + PTIME + $\mathcal{R}$

$\mathcal{R}$ is a second order bounded recursion on notation:

$$\mathcal{R}(x_0, F, B, x) = \begin{cases} x_0 \text{ if } x = 0 \\ t \text{ if } |t| \leq B(t) \\ B(t) \text{ otherwise.} \end{cases}$$

$$\text{with } t = F(x, \mathcal{R}(x_0, F, B, \lfloor \frac{x}{2} \rfloor))$$

BFF: functionals computed by closed PV^ω terms.

Example (Irwin, Kapron, Royer)

$$f_x(y) = 1 \iff y = 2^x$$

$$\Phi, \Psi : \overbrace{((\mathbb{N} \to \mathbb{N}) \to \mathbb{N})}^{F} \times \overset{x}{\mathbb{N}} \to \mathbb{N}$$

$$\Phi(F, x) = \begin{cases} 0 & \text{if } F(f_x) = F(f_\infty) \\ 1 & \text{otherwise.} \end{cases}$$

$\Phi \in \mathrm{BFF}$

$$\Psi(F, x) = \begin{cases} 0 & \text{if } F(f_x) = F(f_\infty) \\ 2^x & \text{otherwise.} \end{cases}$$

$\Psi \notin \mathrm{BFF}$

The size issue

Definition (Output size)

If $F : \tau_1 \times \cdots \times \tau_n \to \mathbb{N}$, then $|F| : \tau_1 \times \cdots \times \tau_n \to \mathbb{N}$ and:

$$|F|(t) = \max_{|f| \leq t} |F(f)|$$

Theorem

The output size of every BFF *functional is well-defined.*

The size issue

Example

$$\Gamma(F) = \begin{cases} 0 \text{ if } \forall x, F(f_\infty) = F(f_x) \\ x \text{ minimal such that } F(f_\infty) \neq F(f_x) \text{ otherwise.} \end{cases}$$

$$\forall x, F_x(f) = \begin{cases} 1 \text{ if } f(x) = 1 \\ 0 \text{ otherwise} \end{cases}$$

$$\forall x, |F_x| \leq \mathbf{1}$$

$\Gamma \notin$ BFF since $|\Gamma(F_x)|$ is unbounded while $|F_x|$ is bounded.

Toward a machine model at higher types

Interaction with the argument in an Oracle Turing Machine:

- Machine: what is $f(n)$?
- Oracle: $f(n)$ is v!

Toward a machine model at higher types

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- Oracle: what is x?
- Machine: $x = n$!
- Oracle: $f(x) = v$!

Toward a machine model at higher types

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- Oracle: what is x?
- Machine: $x = n$!
- Oracle: $f(x) = v$!

Let's generalize this dialogue to all types: a functional is described by the way it interacts with input functionals.

- We first define dialogs as games following strategies.
- We then define HOTM playing such games.

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Higher order strategies

(Hyland & Ong, Nickau)

Finite types: $\tau = \mathbb{N} \mid \tau_1 \times \ldots \times \tau_n \to \mathbb{N}$

Given a finite type, give a name to each occurrence of $\mathbb{N}$:

$$\overbrace{(\underbrace{(\overset{x}{\mathbb{N}} \to \overset{f}{\mathbb{N}}) \to \overset{F}{\mathbb{N}})}_{\text{Opponent}} \to \overset{\phi}{\mathbb{N}}}^{\text{Player}}$$

Moves: $?^f$ or $!^f(v)$.

Strategy: partial function which given a list of previous moves, outputs a valid move.

Execution tree: tree representation of a strategy.

Examples

- $x = 3$

$$\frac{?^x}{} \; !^x(3)$$

Examples

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- $f(x) = 2x + 1$

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$$\frac{?^f}{\quad} \; ?^x \; \frac{!^x(n)}{\quad} \; !^f(2n+1)$$

Examples

- $x = 3$
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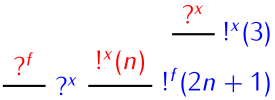
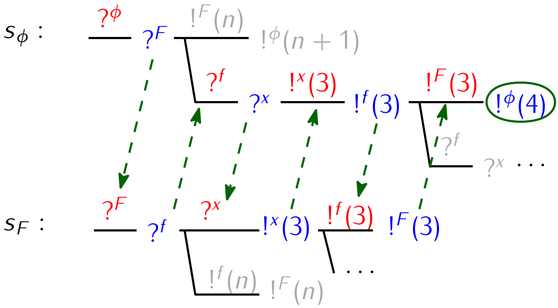
Examples

- $x = 3$
- $f(x) = 2x + 1$
- $((\overset{x}{\mathbb{N}} \rightarrow \overset{f}{\mathbb{N}}) \rightarrow \overset{F}{\mathbb{N}}) \rightarrow \overset{\phi}{\mathbb{N}}$





$\phi(F) = F(\lambda x.x) + 1$

$F(f) = f(3)$

$s_\phi[s_F] = 4$

Representation of functionals by strategies

Definition

$s_F[s_1, \ldots, s_n] = v$ if the game between s_F and $s_1, \ldots, s_n$ ends with $!_F(v)$.

Remark

A game may not end if:

- at some point a strategy is undefined
- or the dialogue is infinite

Definition

s_F represents $F : \tau_1 \times \cdots \times \tau_n \to \mathbb{N}$ if whenever $s_1, \ldots s_n$ represent $f_1, \ldots f_n, s_F[s_1, \ldots, s_n] = F(f_1, \ldots f_n)$.

Computability and continuity

Definition

- A strategy is computable if it is computable as an order 1 function.
- A function is computable if it is represented by a computable strategy.

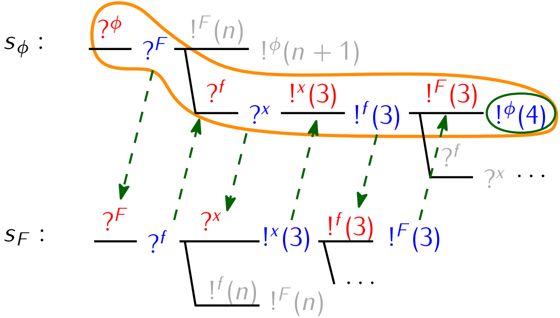
Proposition

A function is (Kleene-Kreisel) continuous if and only if it is represented by a strategy.

Proposition

A function is computable if and only if it is Kleene-Kreisel computable.

Size of a strategy



History: $H(s_\phi, s_F)$

Size of a strategy

Definition (Size)

By induction, the size of a strategy s over type τ is $S_s : \tau$

- If $s = \xrightarrow{?^x} !^x(n)$ then $S_s = |n| + c$.
- $S_s(b_1, \ldots, b_n) = \max_{(s_1, \ldots s_n) \in K_{b_1} \times \cdots \times K_{b_n}} |H(s, s_1, \ldots s_n)|$ with $K_b = \{s' \mid S_{s'} \preccurlyeq b\}$

Example

- $n \in \mathbb{N}$ has a strategy of size $\mathcal{O}(\log_2 n)$.
- $f : \mathbb{N} \to \mathbb{N}$ has a strategy of size $|f|(n) = n + \max_{|x| \leq n} |f(x)|$.
- The size of a strategy for $F : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ is at least its modulus of continuity.

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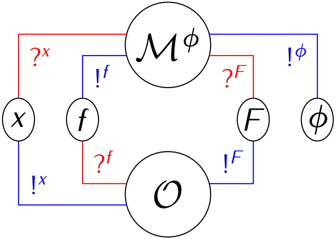
Definition (HOTM)

A HOTM is a kind of oracle Turing machine which plays a game versus strategies played by oracles.

If $\mathcal{M}^\phi$ computes

$\phi : ((\overset{x}{\mathbb{N}} \to \overset{f}{\mathbb{N}}) \to \overset{F}{\mathbb{N}}) \to \overset{\phi}{\mathbb{N}}$ then $\mathcal{M}^\phi$ has four special states denoted by "x","f","F","ϕ".



Running time of a HOTM: same as for an OTM.

Property

A strategy is computable $\iff$ it is represented by a HOTM.

Polynomial time complexity

Definition (Higher type polynomials)

HTP: simply-typed $\lambda-$calculus with $+, * : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Property

HTP *of type* 1 *and* 2 *are respectively the usual polynomials and the second-order polynomials.*

Definition (POLY)

$\phi \in$ POLY if ϕ is computed by a HOTM whose running time is bounded by a HTP.

Remark

- $\text{POLY}_1 = \text{FPTIME}$, $\text{POLY}_2 = \text{BFF}_2$ and $\forall i \geq 3, \text{BFF}_i \subsetneq \text{POLY}_i$.
- We can define other time (or space) complexity classes

Example

$$\Psi(F,x) = \begin{cases} 0 & \text{if } F(f_x) = F(f_\infty) \\ 2^x & \text{otherwise.} \end{cases}$$

The complexity of Ψ is about $\mathcal{F}, n \mapsto c \times \mathcal{F}(P(n))$, where $P(|x|)$ is the complexity of f_x.

Example

$$\Gamma(F) = \begin{cases} 0 \text{ if } \forall x, F(f_\infty) = F(f_x) \\ x \text{ minimal such that } F(f_\infty) \neq F(f_x) \text{ otherwise.} \end{cases}$$

The complexity of Γ is about $\mathcal{F} \mapsto \mathcal{F}(P_\infty) \times \mathcal{F}(P(\mathcal{F}(P_\infty)))$ where P_∞ is the complexity of f_∞.

Higher order polynomial time complexity

- ✓ Inputs: strategies
- ✓ Size of inputs
- ✓ Machine model
- ✓ Running time
- ✓ Polynomial time complexity class

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Complexity in computable analysis

- Order 1
  - Complexity of real functions (Ko, Friedman)
  - Generalization to σ-compact spaces (Weihrauch, Schröder)
- Order 2 (Kawamura and Cook)
  - Polynomial time complexity based on BFF_2
  - Allows to define notions of complexity over non σ-compact spaces like $\mathcal{C}([0,1],\mathbb{R})$
- Is order 2 always sufficient?

"Feasible" admissibility

Definition (Polynomial reducibility)

$\delta \leq_P \delta'$ if $\delta = \delta' \circ f$ with f polynomial time computable

Theorem (Kawamura & Cook)

$\delta_\square$ is the "largest" representation of $\mathcal{C}([0,1],\mathbb{R})$ making $Eval : \mathcal{C}([0,1],\mathbb{R}) \to [0,1] \to \mathbb{R}$ polynomial time computable.

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$\to$ For which spaces can we do the same?

Question (Kawamura)

For which spaces X the space $\mathcal{C}(X,\mathbb{R})$ admits a (maximal) representation making $Eval : \mathcal{C}(X,\mathbb{R}) \times X \to \mathbb{R}$ polynomial time computable?

First order representations are not sufficient

Theorem

Let X be a Polish space that is not σ-compact. Then there is no representation of $\mathcal{C}(X, \mathbb{R})$ making the time complexity of $Eval_{X,\mathbb{R}} : \mathcal{C}(X, \mathbb{R}) \times X \to \mathbb{R}$ well-defined.

($X = \mathcal{C}([0, 1], \mathbb{R})$ for example)

Lemma

There is no surjective partial continuous function $\phi : (\mathbb{N} \to \mathbb{N}) \to \mathcal{C}(\mathbb{N} \to \mathbb{N}, \mathbb{N})$ bounded by a total continuous function.

Corollary

"Higher order is required to define complexity-friendly representations."

Higher order representations

Definition (Kleene-Kreisel Spaces)

$KKS = [\mathbb{N}, \subseteq, \rightarrow, \times]$

Definition (Representation)

A representation δ of a space X with a KKS A is a surjective function from A to X.

Definition (Polynomial reduction)

$\delta_1 \leq_P \delta_2$ if $\delta_1 = \delta_2 \circ F$ for some polynomial time computable $F : A_1 \rightarrow A_2$.

Standard representation of $\mathcal{C}(X, Y)$

Definition

$\delta_{\mathcal{C}(X,Y)}(F) = f$ whenever $f \circ \delta_X = \delta_Y \circ F$

Property

$Eval : \mathcal{C}(X, Y) \times X \to Y$ is polynomial-time computable w.r.t. $(\delta_{\mathcal{C}(X,Y)}, \delta_X, \delta_Y)$

Theorem

It is the largest representation making Eval polynomial.

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- We have a robust definition of higher order complexity.
- This gives us new representation spaces.
- Some spaces can now be well represented.

- We need to understand the boundaries of the class of polynomial time functionals.
- Make further comparisons with BFF.
- Give implicit characterizations (e.g. function algebra like PV^{ω}).
- Study the extension of TTE with these new representations (e.g. admissibility)
- Find applications in other domains.