Infinitary Lambda Calculi from a Linear Viewpoint

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Inria informatiques mathématiques

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Computation over Streams

From $f : \{0,1\}^* \to \{0,1\}^* \ldots$

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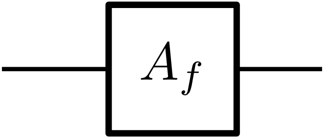

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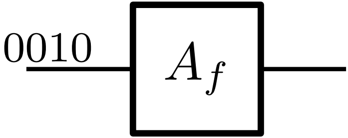

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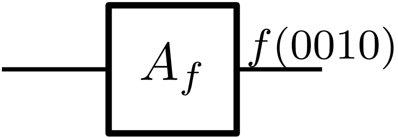

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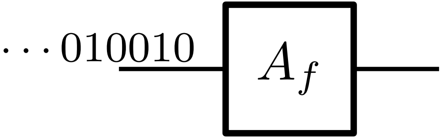

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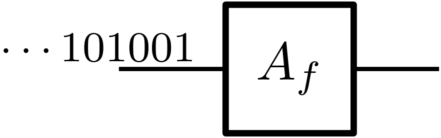

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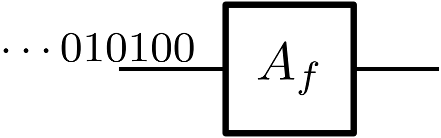

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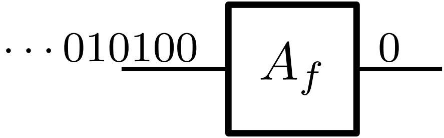

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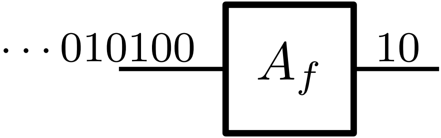

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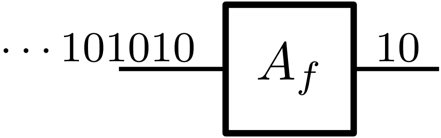

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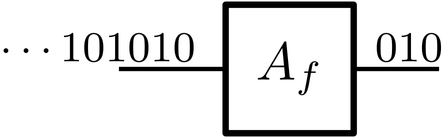

Termination

(A_f produces an output in finite time)

$$\Downarrow$$

Productivity

(A_f produces “the next” bit in finite time)

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Infinitary Term Rewriting – Example

$$\mathit{flip} : \{0,1\}^\omega \to \{0,1\}^\omega.$$

$$\Sigma = \{\mathtt{0}, \mathtt{1}\}$$
$$s \in \{0,1\}^\omega \mapsto \overline{s} \in \mathit{InfTerms}(\Sigma)$$

$$\mathtt{f}(0 \cdot s) \to 1 \cdot \mathtt{f}(s)$$
$$\mathtt{f}(1 \cdot s) \to 0 \cdot \mathtt{f}(s)$$

$$\mathtt{f}(\overline{s}) \longrightarrow^\omega \overline{\mathit{flip}(s)}$$

- Productivity can be checked effectively [EndrullisGrabmayerHendriksIsiharaKlop2010].

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Infinity and the λ-calculus

- It works well in the finitary case (as we all know).
- Already in usual Λ, computation over streams can be modeled through **infinitary interaction** and **laziness**.
  - As we know, $\mathtt{0}$, $\mathtt{s}$ and $\langle\cdot,\cdot\rangle$ can all be defined in Λ.
  - Consider

$$N = \lambda x.\lambda y.\lambda z.\langle y, x(\mathtt{s}y)\rangle;$$
$$M_{\mathtt{n}} = Y\ N\ \mathtt{n}.$$

  - For every $\mathtt{n}$ and for every v, $M_{\mathtt{n}}\ v$ lazily evaluates to $\langle \mathtt{n}, M_{\mathtt{n+1}}\rangle$.
  - As a consequence, $M_{\mathtt{n}}$ can be seen as representing the stream

$$n \cdot (n+1) \cdot (n+2) \cdot (n+3) \cdot \ldots$$

  - But infinitely many interactions with $M_{\mathtt{n}}$ and infinitely many (finite) computations are necessary to "discover" the value it represents!
- We would like infinity to become a **first-class citizen**...

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Infinity and the λ-calculus: Λ_∞ [KSSV1998]

- Terms are **potentially infinite** trees built from the usual grammar:
$$M ::= x \mid \lambda x.M \mid MN$$
- Analogously, one can define the language of terms as the metric completion of Λ with respect to an appropriate metric.
  - Which metric?
  - Two terms differ by $\frac{1}{2^n}$ if they are equal up to **depth** n.
  - Where does the depth **grow**? Whenever we go inside a term?
- Too much infinity:
  - **Productivity** is not guaranteed (expected);
  - **Complete Developments Theorem** does not hold;
  - **Confluence** is lost (but can be recovered in Böhm reduction).

- Depending on **how** you define the depth of a term, you can get $8 = 2^3$ calculi.
- Examples:

$$\Lambda_{001}$$
$$M \equiv \lambda x.xM$$
$$N \equiv \lambda x.N$$

$$\Lambda_{100}$$
$$M \equiv (\lambda x.x)M$$
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Reduction Sequences of Infinite Length

- Suppose
$$M \to M_1 \to M_2 \to \ldots$$
- Is this a **reasonable** reduction sequence? Yes, provided
  - the sequence $\{M_i\}_{i \in \mathbb{N}}$ **converges** to a term N.
  - the sequence $\{d_i\}_{i \in \mathbb{N}}$, where d_i is the depth o of the redex fired at M_i, **tends to** infinity.
- In this case we write $M \Rightarrow^{\omega} N$.
- This can be generalized to reduction sequences of any ordinal length, but:

Theorem (Compression)

If $M \Rightarrow^{\alpha} N$, then $M \Rightarrow^{\omega} N$.

- What does productivity mean here?
  - Hereditary head normal forms!

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Linearity and Termination in Λ

- Suppose that **linearity** holds in a strong sense: $\#FO(M, x) \leq 1$ for every M.
  - **No duplication** is possible;
  - Λ becomes strongly normalizing;
  - Just a tiny fraction of the class of recursive functions can be computed.
- **Reintroduce** duplication: $\ell\Lambda$.
  - Extend Λ with **non-linear** abstractions in the form $\lambda!x.M$;
  - Mark any term N as **duplicable** by a new operator (called a **box**): $!N$;
  - Consider a new reduction rule: $(\lambda!xM.)!N \rightarrow M\{x/N\}$.
- Duplicate **with care**: $\ell\Lambda^{\mathsf{LLL}}$, $\ell\Lambda^{\mathsf{ELL}}$, $\ell\Lambda^{\mathsf{SLL}}$, ...
  - Put some constraint on non-linear abstractions and boxes.
  - E.g. for every $\lambda!x.M$, all the occurrences of x must be in the scope of exactly one box, i.e. $LVL(x, M) = 1$.

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$\ell\Lambda_\infty$

- Start from the terms of $\ell\Lambda$.
- One could define terms of $\ell\Lambda_\infty$ following exactly the same ideas as the ones used to get Λ_∞.
- Instead, let us follow:

Motto

Duplication and infinity are related, and should be treated with the **same** tools.

- Idea:
  - The operator $\uparrow M$ is the **only** place where depth (in the sense of [KKSV1998]) increases;
  - Terms in the form $\uparrow M$ (called **coinductive boxes**) are duplicable.
- Moreover:
  - The term $\downarrow M$ (an **inductive box**) is itself duplicable;
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$\ell\Lambda_\infty$: Terms and Well-Formation Rules

$$M, N ::= x \;\Big|\; MM \;\Big|\; \lambda x.M \;\Big|\; \lambda{\downarrow}x.M \;\Big|\; \lambda{\uparrow}x.M \;\Big|\; {\downarrow}M \;\Big|\; {\uparrow}M$$

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$$\begin{array}{lcl} \lambda x.M & \Rightarrow & x \text{ linear} \\ \lambda{\uparrow}x.M & \Rightarrow & \text{no restrictions} \\ \lambda{\downarrow}x.M & \Rightarrow & \text{no restrictions} \end{array}$$

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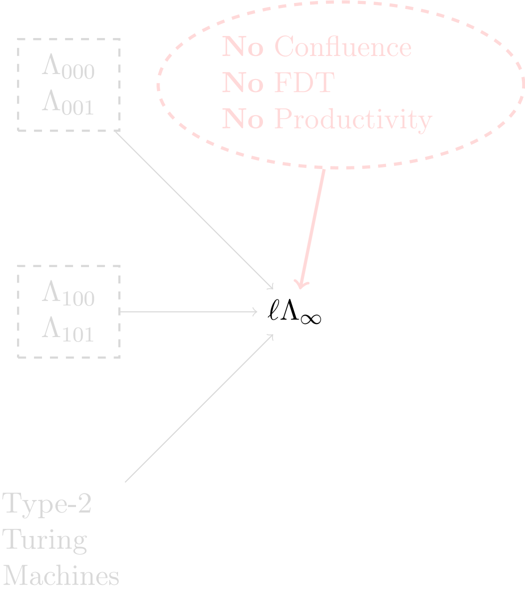
$$\frac{}{{\downarrow}\Theta, {\uparrow}\Xi, x \vdash x}\ (\mathsf{vl}) \qquad \frac{}{{\downarrow}\Theta, {\uparrow}\Xi, {\downarrow}x \vdash x}\ (\mathsf{vi})$$

$$\frac{}{{\downarrow}\Theta, {\uparrow}\Xi, {\uparrow}x \vdash x}\ (\mathsf{vc}) \qquad \frac{\Gamma, {\downarrow}\Theta, {\uparrow}\Xi \vdash M \quad \Delta, {\downarrow}\Theta, {\uparrow}\Xi \vdash N}{\Gamma, \Delta, {\downarrow}\Theta, {\uparrow}\Xi \vdash M\,N}\ (\mathsf{a})$$

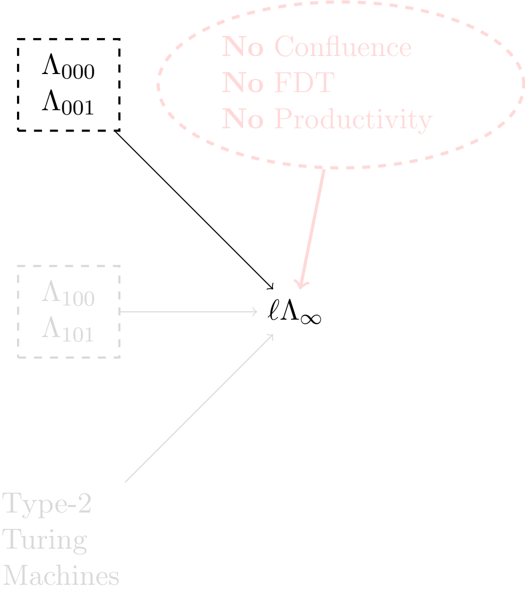
$$\frac{\Gamma, x \vdash M}{\Gamma \vdash \lambda x.M}\ (\mathsf{ll}) \qquad \frac{\Gamma, {\downarrow}x \vdash M}{\Gamma \vdash \lambda{\downarrow}x.M}\ (\mathsf{li}) \qquad \frac{\Gamma, {\uparrow}x \vdash M}{\Gamma \vdash \lambda{\uparrow}x.M}\ (\mathsf{lc})$$

$$\frac{{\downarrow}\Theta, {\uparrow}\Xi \vdash M}{{\downarrow}\Theta, {\uparrow}\Xi \vdash {\downarrow}M}\ (\mathsf{mi}) \qquad \genfrac{}{}{1.5pt}{}{{\downarrow}\Theta, {\uparrow}\Xi \vdash M}{{\downarrow}\Theta, {\uparrow}\Xi \vdash {\uparrow}M}\ (\mathsf{mc})$$

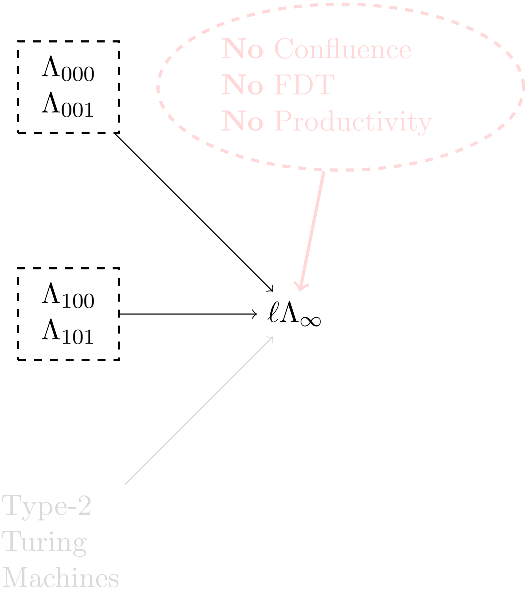
$\ell\Lambda_\infty$: Embeddings



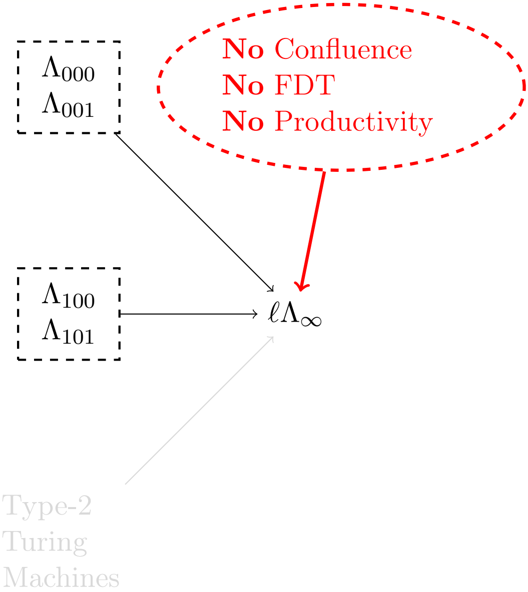
$\ell\Lambda_\infty$: Embeddings



$\ell\Lambda_\infty$: Embeddings



$\ell\Lambda_\infty$: Embeddings



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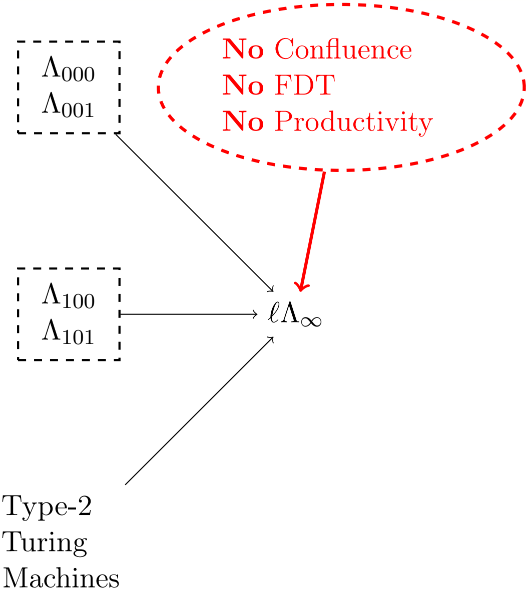
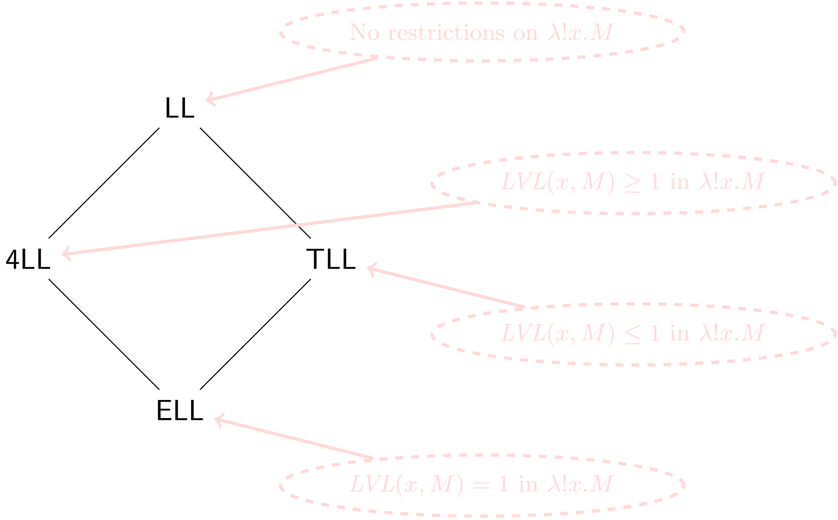


So What?

Drawing Inspiration from Light Logics [DanosJoinet1999]



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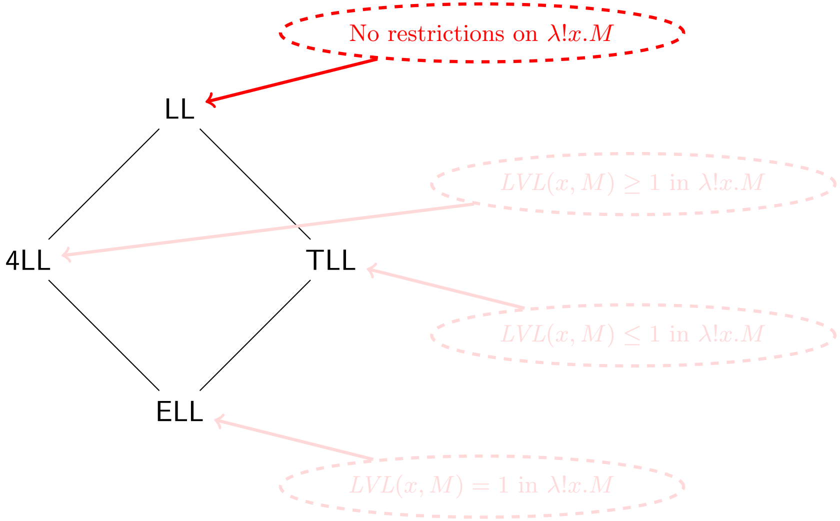

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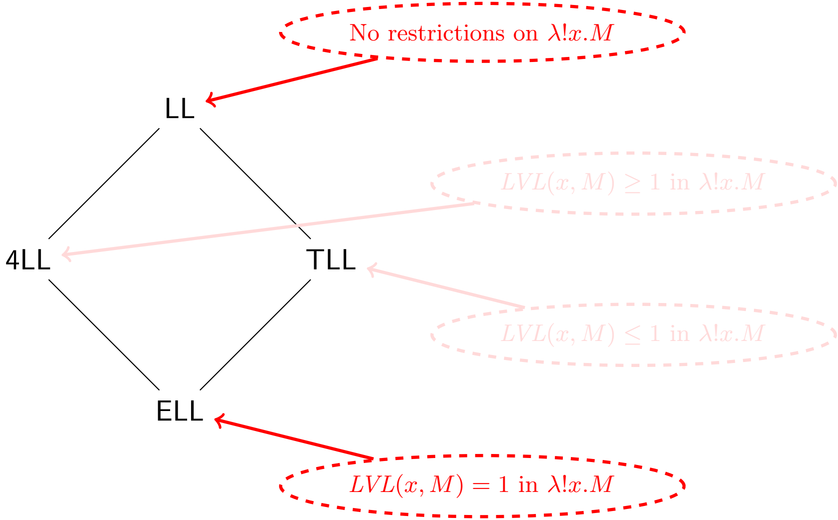

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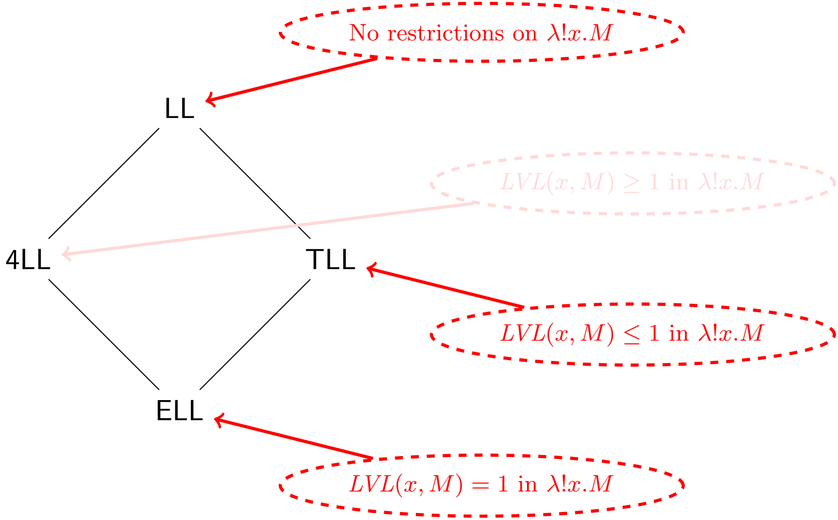

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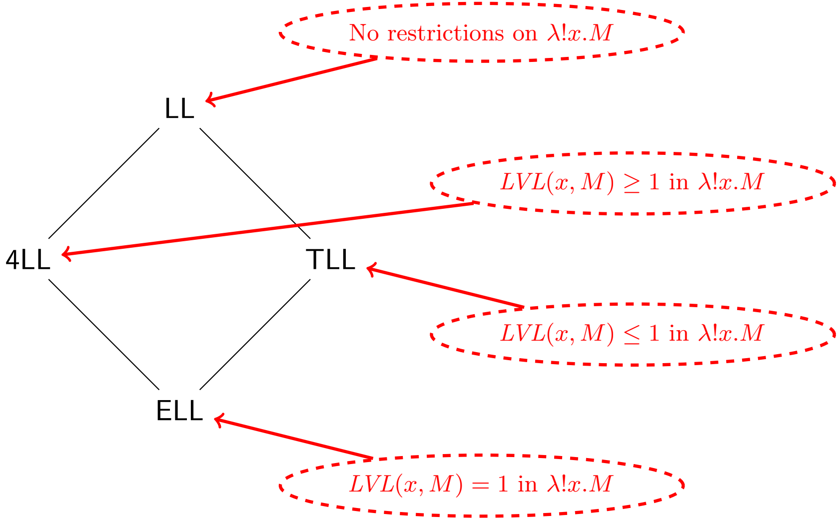

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TLL	4LL
$(\lambda !x.x!x)!(\lambda !x.x!x)$	$(\lambda !x.!(x!x))!(\lambda !x.!(x!x))$
$\downarrow$	$\downarrow$
$(\lambda !x.x!x)!(\lambda !x.x!x)$	$!((\lambda !x.!(x!x))!(\lambda !x.!(x!x)))$
$\downarrow$	$\downarrow$
$\vdots$	$!!((\lambda !x.!(x!x))!(\lambda !x.!(x!x)))$
	$\downarrow$
	$\vdots$

$\ell\Lambda_\infty^{\mathsf{4S}}$: Design Principles

- **Coinductive boxes** should be treated as in $\mathsf{4LL}$.
  - For every $\lambda{\uparrow}x.M$, any occurrence of x appears in the scope of **at least one** $\uparrow$ operator.
- **Inductive boxes** should rather be constrained in such a way as to guarantee termination at each depth.
  - We can grab, as an example, the exponential discipline of SLL [Lafont2004].
  - For every $\lambda{\downarrow}x.M$,
    - either any occurrence of x in M appears outside the scope of any boxes.
    - or there is 1 occurrence of x in M, in the scope of exactly one $\downarrow$ operator.
- The rest of the calculus is the same as in $\ell\Lambda_\infty$.

$\ell\Lambda_{\infty}^{4S}$: Well-Formation Rules

$$\frac{}{\#\Theta, \uparrow\Xi, \updownarrow\Psi, x \vdash x}\ (\mathsf{vl}) \qquad \frac{}{\#\Theta, \uparrow\Xi, \updownarrow\Psi, \#x \vdash x}\ (\mathsf{vd}) \qquad \frac{}{\updownarrow\Theta, \uparrow\Xi, \updownarrow\Psi, \updownarrow x \vdash x}\ (\mathsf{va})$$

$$\frac{\Upsilon, \#\Theta, \uparrow\Xi, \updownarrow\Psi \vdash M \quad \Pi, \#\Theta, \uparrow\Xi, \updownarrow\Psi \vdash N}{\Upsilon, \Pi, \#\Theta, \uparrow\Xi, \updownarrow\Psi \vdash MN}\ (\mathsf{a}) \qquad \frac{\Gamma, x \vdash M}{\Gamma \vdash \lambda x.M}\ (\mathsf{ll})$$

$$\frac{\Gamma, \#x \vdash M}{\Gamma \vdash \lambda{\downarrow}x.M}\ (\mathsf{li})_1 \qquad \frac{\Gamma, \downarrow x \vdash M}{\Gamma \vdash \lambda{\downarrow}x.M}\ (\mathsf{li})_2 \qquad \frac{\Gamma, \uparrow x \vdash M}{\Gamma \vdash \lambda{\uparrow}x.M}\ (\mathsf{lc})$$

$$\frac{\Xi, \uparrow\Psi, \updownarrow\Phi \vdash M}{\#\Theta, \downarrow\Xi, \uparrow\Psi, \updownarrow\Phi \vdash \downarrow M}\ (\mathsf{mi}) \qquad \frac{\updownarrow\Xi, \updownarrow\Psi \vdash M}{\#\Theta, \uparrow\Xi, \updownarrow\Psi \vdash \uparrow M}\ (\mathsf{mc})$$

$\ell\Lambda_{\infty}^{\mathsf{4S}}$: Main Results

Lemma (Productivity)

For every $n \in \mathbb{N}$, the relation $\to_n$ is strongly normalizing.

Theorem (Depth-by-depth Normalization)

For every term M there is a normal form N such that $M \Longrightarrow N$.

Theorem (Complete Developments Theorem)

For every M and for every set $\mathcal{R}$ of redexes in M, there is a complete development of $\mathcal{R}$.

Theorem (Confluence)

If M, $M \Longrightarrow N$ and $M \Longrightarrow L$, then there is $P \in \ell\Lambda_{\infty}^{\mathsf{4S}}$ such that $N \Longrightarrow P$ and $L \Longrightarrow P$.

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Theorem (Causality)

If $M \equiv_n N$, $M \Longrightarrow L$ and $N \Longrightarrow P$ with L, P normal forms, then $L \equiv_n P$.

Theorem

Primitive co-recursion can be embedded into $\ell\Lambda_{\infty}^{\mathsf{4S}}$.

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Theorem

Primitive co-recursion can be embedded into $\ell\Lambda_{\infty}^{\mathsf{4S}}$.

Future Work

- **Complexity**
  - What is the class of functions (on streams) that can be captured by $\ell\Lambda_\infty^{\mathsf{4S}}$?
  - Is there a restriction of $\ell\Lambda_\infty$ capturing one of the notions of polynomial time from the literature (e.g. [KawamuraCook2012])?
- **Semantics**
  - Relational semantics?
  - Game semantics with infinite, but total strategies?
  - Ultra-metric spaces?
- **Types**
  - Recursive Types?
  - Linear Dependent Types [DLGaboardi2011]?

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Thank you!

Questions?